

1. Determine the critical points of the function and find the intervals on which the function is increasing and decreasing. Use this information to find relative maxima and relative minima of the function.

a) $f(x) = x^3 - 3x + 8$

Solution. $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. Solve $3(x^2 - 1) = 0$ to find the critical points at $x = -1$ and $x = 1$. Checking the sign of $f'(x)$ we find that f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

b) $f(x) = x^5 - (5/2)x^4 - 10$

Solution. $f'(x) = 5x^4 - 10x^3 = 5x^3(x - 2)$. Solve $5x^3(x - 2) = 0$ to find the critical points at $x = 0$ and $x = 2$. Checking the sign of $f'(x)$ we find that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$.

c) $f(x) = x^5 - 5x^4 + \left(\frac{20}{3}\right)x^3 - 5$

Solution. $f'(x) = 5x^4 - 20x^3 + 20x^2 = 5x^2(x^2 - 4x + 4) = 5x^2(x - 2)^2$. Solve $5x^2(x - 2)^2 = 0$ to find the critical points at $x = 0$ and $x = 2$. Notice that $f'(x)$ is always greater than or equal to 0. Thus f has no relative extrema.

d) $f(x) = x^{\frac{2}{3}} + 2$.

Solution. $f'(x) = \frac{2}{3x^{\frac{1}{3}}}$. This means that $f'(x)$ is never 0. However, $f'(x)$ is undefined at $x = 0$ and 0 is in the domain of f . Thus we have a critical point at $x = 0$. Checking the sign of $f'(x)$ shows that the function f has a relative minimum at $x = 0$.

e) $f(x) = \frac{x}{x+1}$

Solution. Notice that the domain of f is $(-\infty, -1) \cup (-1, \infty)$. Use the quotient rule to find

$$f'(x) = \frac{x + 1 - x}{(x + 1)^2} = \frac{1}{(x + 1)^2}.$$

As in part d, $f'(x)$ is never zero but $f'(x)$ is undefined at $x = -1$. This time, however, -1 is not in the domain of f , so there are no critical points.

2. Find $\frac{dy}{dx}$.

a) $2x^3 - 3xy = 4$

Solution. Differentiate both sides with respect to x to find $6x^2 - 3\left(y + x\frac{dy}{dx}\right) = 0$. Solve for $\frac{dy}{dx}$ to find

$$\frac{dy}{dx} = \frac{3y - 6x^2}{-3x} = \frac{2x^2 - y}{x}.$$

b) $x^2 + 2x^2y^2 + y^2 = 10$

Solution. Differentiate both sides with respect to x to find $2x + 2\left(2xy^2 + 2x^2y\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$.

Solve for $\frac{dy}{dx}$ to find

$$\frac{dy}{dx} = \frac{-2x - 4xy^2}{2y + 4x^2y} = \frac{-x - 2xy^2}{y + 2x^2y}.$$

3. Find the second derivative of the function.

a) $f(x) = \frac{1}{x^2+1}$

Solution. $f'(x) = \frac{-2x}{(x^2+1)^2}$ so we must use the quotient rule to find the second derivative.

$$f^{(2)}(x) = \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} = \frac{6x^2-2}{(x^2+1)^3}$$

b) $g(t) = t(t^2+1)^3$.

Solution. Use the product rule (or multiply out the cubic in the original function).

$$g^{(2)}(t) = 18t(t^2+1)^2 + 24t^3(t^2+1) = 42t^5 + 60t^3 + 18t$$

4. Show that if $f(x) + g(y) = 0$ and f and g are differentiable, then $\frac{dy}{dx} = -\frac{f'(x)}{g'(y)}$.

Solution. Differentiate both sides of the equation $f(x) + g(y) = 0$ with respect to x , being sure to apply the chain rule. The result is $f'(x) + \frac{dy}{dx}g'(y) = 0$. Solve for $\frac{dy}{dx}$ to get $\frac{dy}{dx} = -\frac{f'(x)}{g'(y)}$.

5. A spherical bubble is being filled with air at a rate of $8 \text{ cm}^3/\text{s}$. How fast is the radius of the bubble increasing when $r = 3 \text{ cm}$? (The volume of a sphere is $V = \frac{4}{3}\pi r^3$).

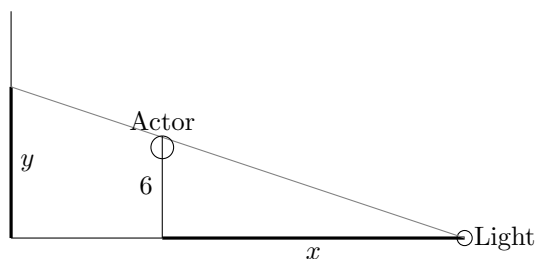
Solution. First differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t . The result is

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given that $\frac{dV}{dt} = 8$ and $r = 3$. Substituting in these quantities and solving for $\frac{dr}{dt}$ gives

$$\frac{dr}{dt} = \frac{2}{9\pi}.$$

6. An actor on a stage walks toward a foot-light at the front of the stage at a rate of 3 ft/s . The foot-light is 30 ft from the back of the stage and the actor is 6 ft tall. How fast is the height of the actor's shadow growing? Hint: use similar triangles.



Solution. The height of the shadow is y and the distance from the actor to the light is x . We are given that $\frac{dx}{dt} = -3 \text{ ft/s}$ (x shrinks at a rate of 3 ft/s) and we wish to find $\frac{dy}{dt}$. Properties of similar triangles give the relationship between x and y :

$$\frac{y}{6} = \frac{30}{x}.$$

This is easiest to use if we move the variables to one side to get $xy = 180$. Implicit differentiation gives

$$\frac{dy}{dt} = \frac{-\frac{dx}{dt}y}{x} = \frac{3y}{x}.$$

We are not given a specific value for x . However we can use the solution to calculate that when $x = 30$, the shadow is 6 ft tall and $\frac{dy}{dt} = 0.6 \text{ ft/s}$. When $x = 15$, the shadow is 12 ft tall and $\frac{dy}{dx} = 2.4 \text{ ft/s}$.