148 Exam 3 Review.

1. a) • $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$.

$f'(x)$

$f'(-2) = 3(-3)(-1) > 0$
$f'(0) = 3(-1)(1) < 0$
$f'(2) = 3(1)(3) > 0$

f is increasing on $(-\infty, -1)$ and on $(1, \infty)$.
f is decreasing on $(-1, 1)$.

• $g'(x) = \dfrac{-2}{(2x+3)^2}$ (use chain rule or quotient rule)

$g'(x) < 0$ for all x because the denominator is always non-negative while the numerator is always negative.
$x = -\frac{3}{2}$ is not in the domain of g (g has a vertical asymptote at $x = -\frac{3}{2}$).
g is decreasing on $(-\infty, -\frac{3}{2})$ and on $(-\frac{3}{2}, \infty)$

b) • $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$, this is the same derivative as the first problem above. Use the above work.
f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

• $g'(x) = \dfrac{x+1-x}{(x+1)^2} = \dfrac{1}{(x+1)^2}$

$g'(x)$ never equals 0 but it is undefined at $x = -1$.
So $x = -1$ is a critical point.
Because of the square in the denominator $g'(x)$ will never be negative. Therefore g has no relative extrema.

2. a) ▸ $f'(x) = 3x^2 - 1$

$f''(x) = 6x$

$6x = 0$ when $x = 0$

Inflection point at $x = 0$.

f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

▸ $g'(x) = \dfrac{1-x-(-1)x}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

$g''(x) = \dfrac{-2(-1)}{(1-x)^3} = \dfrac{2}{(1-x)^3}$ using the chain rule,

$g''(x)$ is never 0, so the graph of g has no inflection points.

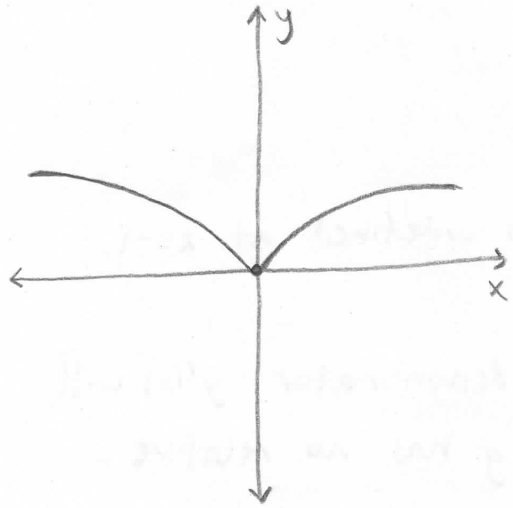
However, $g''(x) > 0$ on $(-\infty, 1)$ and $g''(x) < 0$ on $(1, \infty)$. Therefore the graph of g is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$.

b) ▸ $f^{(2)}(0) = -1 < 0$ so f has a relative maximum at $x = 0$

$f^{(2)}(1) = 1 > 0$ so f has a relative minimum at $x = 1$

▸ $h^{(2)}(0) = -\dfrac{6(-3)}{3^4} > 0$ so h has a relative minimum at $x = 0$.

c)



3. a) $f(x) = \frac{x+1}{2x-1}$. Undefined at $x = \frac{1}{2}$. $\lim_{x \to \frac{1}{2}^+} f(x) = \infty$ so $x = \frac{1}{2}$ is a vertical asymptote.

$\lim_{x \to \infty} \frac{x+1}{2x-1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{2 - \frac{1}{x}} = \frac{1}{2}$. Therefore $y = \frac{1}{2}$ is a horizontal asymptote.

$\lim_{x \to -\infty} \frac{x+1}{2x-1} = \lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{2 - \frac{1}{x}} = \frac{1}{2}$. Same horizontal asymptote as before.

b) Horizontal asymptote at $y=0$.
Vertical asymptotes at $x = -2$ and $x = 1$.

c) Horizontal asymptote at $y = 1$
Vertical asymptote at $x = 3$

d) $r(x) = \frac{1-x^2}{x^2+x} = \frac{(1-x)(1+x)}{x(x+1)} = \frac{1-x}{x}$ when $x \neq -1$.

Thus r has a vertical asymptote at $x=0$ and a horizontal asymptote at $y = -1$.

4. a) $f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}}$. $f'(x)$ never equals 0 but is undefined at $x = 0$. $f(0)$ is defined so $x=0$ is a critical point for f. Use the first derivative test.

$$\xleftarrow{\quad - \quad} \overset{?}{\underset{0}{|}} \xrightarrow{\quad + \quad}$$

$f'(x)$

$f'(-1) = \frac{2}{3(-1)^{1/3}} = -\frac{2}{3} < 0$

$f'(1) = \frac{2}{3(1)^{1/3}} = \frac{2}{3} > 0$

Therefore $f(0) = 0$ is a relative minimum. In fact, since f has no other critical points $f(0) = 0$ is the absolute minimum.

b) $g'(x) = \frac{1}{(1-x)^2}$. $g'(x)$ is never 0 and is undefined only where $g(x)$ is also undefined. Thus g has no critical points.

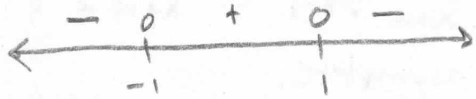
$g(2) = -2$ and $g(6) = -\frac{6}{5}$. Therefore the absolute maximum for $g(x)$ over $[2, 6]$ is $-\frac{6}{5}$, reached when $x = 6$.

4. continued.

d) $f(x)=\dfrac{x}{x^2+1}$. $\quad f'(x)=\dfrac{x^2+1-x(2x)}{(x^2+1)^2}=\dfrac{1-x^2}{(x^2+1)^2}$.

$f'(x)=0$ when $x=1$ and when $x=-1$.

Use the first derivative test.



$f'(x)$

$f'(-2)=\dfrac{1-4}{(4+1)^2}<0$

$f'(0)=\dfrac{1}{1^2}>0$

$f'(2)=\dfrac{1-4}{(4+1)^2}<0$

Thus $x=1$ gives a relative maximum and $x=-1$ gives a relative minimum. A quick glance at the graph of the function verifies that these are absolute extrema.

5. b) Revenue function: $R(x)=px-C(x)=(-.00042x+6)x-(600+2x-.00002x^2)$

$$=-600+4x-.0004x^2.$$

Optimizing as usual shows that there is an absolute maximum value of 9400 at $x=5000$.

c) Volume: $x^2y=108$. Hence $y=\frac{108}{x^2}$

Material used is given by surface area: $M=x^2+4xy$

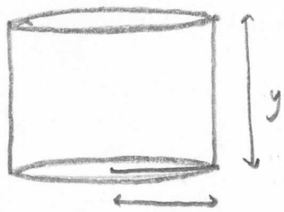
$$=x^2+4x\left(\frac{108}{x^2}\right)$$

$$=x^2+\frac{432}{x}$$

Optimizing as usual we find a relative minimum at $x=6$. This turns out to be an absolute minimum over the domain $(0,\infty)$ as we see by graphing the function or using the first derivative test. Therefore the minimum amount of material that can be used is 108 in² and this is achieved when $x=6$ and $y=3$.

5. continued.

d) Volume: $36 = \pi r^2 y$ so $y = \frac{36}{\pi r^2}$



surface area: $2\pi r^2 + 2\pi r y$

Substitute in $y = \frac{36}{\pi r^2}$ to get

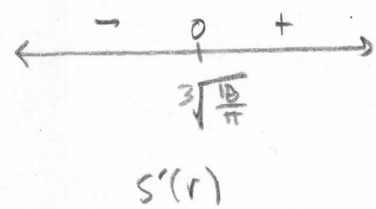
$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{36}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{72}{r}$$

Optimize as usual: $S'(r) = 4\pi r - \frac{72}{r^2}$

$$0 = 4\pi r - \frac{72}{r^2} \Leftrightarrow 72 = 4\pi r^3 \Leftrightarrow r = \sqrt[3]{\frac{18}{\pi}}$$

First derivative test:

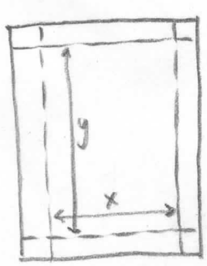


$S'(1) = 4\pi - 72 < 0$

$S'(10) = 40\pi - \frac{72}{100} > 0$

Therefore $r = \sqrt[3]{\frac{18}{\pi}}$ gives an absolute minimum surface area of $2\pi\left(\sqrt[3]{\frac{18}{\pi}}\right)^2 + \frac{72}{\sqrt[3]{\frac{18}{\pi}}} \approx 60.3554 \text{ in}^2$.

f)



Printed area: $54 = xy$ so $y = \frac{54}{x}$

Area of a page: $A = (x+1.5)(y+1)$

$$= xy + 1.5y + x + 1.5$$

$$= 54 + \frac{81}{x} + x + 1.5$$

$A'(x) = 1 - \frac{81}{x^2}$.

$0 = 1 - \frac{81}{x^2} \Leftrightarrow x^2 = 81 \Leftrightarrow x = \pm 9$

$A''(x) = \frac{144}{x^3}$ so $A''(9) > 0$, meaning that $x = 9$ gives a minimum area for the page. That area is $(9+1.5)(6+1) = 73.5$