4:30 Wed. Review

Limits and continuity including asymptotes.

1. Find the following limits, if they exist.

a) $\lim_{x\to 1} \frac{x-1}{x} = 0$

b) $\lim_{x\to 0} \frac{x-1}{x}$ DNE

c) $\lim_{x\to -2} \frac{x^2-x-6}{x^2-4} = \lim_{x\to -2} \frac{(x-3)(x+2)}{(x-2)(x+2)} = \frac{-5}{-4} = \frac{5}{4}$

2. Determine if the function is continuous on the given interval.

a) $f(x) = \frac{1}{x-1}$ over $[-1, 0]$ f(x) is undefined when x=1, continuous everywhere else

b) $f(x) = \frac{x^2-x-6}{x^2-4}$ over $[0, 4]$ f(x) is undefined when $x = \pm 2$, continuous everywhere else

3. Identify the horizontal and vertical asymptotes of the function.

	Horizontal	Vertical
a) $f(x) = \frac{x}{x^2-4}$	$y=0$	$x=-2, x=2$
b) $f(x) = \frac{3x-1}{2x+4}$	$y=\frac{3}{2}$	$x=-2$

The definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

4. Evaluate the limit $\lim_{h\to 0} \frac{\ln(1+h)}{h} = \frac{d}{dx}\ln x \Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$

Differentiation. including the basic rules of section 3.1, the product rule, the quotient rule, and the chain rule.

5. Differentiate the function.

a) $f(x) = (3x-1)^5(x+2)^2$ $f'(x) = 15(3x-1)^4(x+2)^2 + 2(3x-1)^5(x+2)$

b) $f(x) = \frac{1}{5x^2-x}$ $f'(x) = -(10x-1)(5x^2-1)^{-2}$

c) $f(x) = \frac{(x^2-1)^3}{\sqrt{x^2+4}}$ $f'(x) = \left[6x(x^2-1)^2\sqrt{x^2+4} - (x^2-1)^3 x (x^2+4)^{-\frac{1}{2}}\right](x^2+4)^{-1}$

Implicit differentiation and related rates.

6. Find $\frac{dy}{dx}$ by implicit differentiation when $x^2 - xy^2 - 2y^4 = 13$.

7. A cylindrical coffee pot with a radius of 4 in. is being filled so that the level of coffee is rising at a rate of 0.4 in./sec. At what rate is coffee flowing into the coffee pot? (The volume of a cylinder of height h and radius r is $\pi r^2 h$).

Applications of the first and second derivative. Slopes of tangent lines. The first derivative and increasing and decreasing functions. The second derivative and concavity.

8. Find the slope of the tangent line to the curve $(x-1)^2 + y^2 = 5$ at the point $(0, 2)$.

9. Determine the intervals on which the function $f(x) = \frac{1}{x^2+1}$ is concave up and concave down.

Optimization. Relative and absolute extrema and the first and second derivative tests.

10. Find the absolute extrema of the function $f(x) = x^3 + x^2 - 2x$ over $[-2, 2]$.

11. Find any relative extrema of the function with derivative $f'(x) = \frac{x}{x^2+1}$ and determine if each is a minimum or maximum.

12. An open box has a square base and a volume of 108in.^{3}. Find the dimensions that minimize the amount of material used to construct the box.

Working with exponential and logarithmic functions. Especially the laws of exponents on P. 331 and the laws of logarithms on P. 339.

13. Simplify the following expressions.

a) $\ln\left[\frac{xe^x}{(x-1)^4}\right] = \ln(xe^x) - \ln[(x-1)^4] = \ln x + x - 4\ln(x-1)$

b) $\left(e^{x+1}\right)^{\ln 2} = \left(e^{\ln 2}\right)^{(x+1)} = 2^{x+1}$

Compound interest and exponential decay.

14. How much money was initially deposited in an account paying 6% annual interest, compounded monthly, if after 6 years the account is now worth \$8,000?

15. Potassium-40 decays with a half life of 1.3 billion years. How old is a rock sample which contains 32% of its original potassium-40?

Differentiating exponential and logarithmic functions. Including logarithmic differentiation.

16. Find the derivative of the function.

a) $f(x) = xe^{\frac{1}{x+1}}$ $\quad f'(x) = e^{\frac{1}{x+1}} + x\left[\frac{-1}{(x+1)^2}\right]e^{\frac{1}{x+1}}$

b) $f(x) = \ln\left(\frac{1}{x^2}\right)$ $\quad f'(x) = -2x^{-3}x^2 = -2x^{-1}$ or simplify to $f(x) = -2\ln x$

c) $f(x) = \ln\sqrt{x^3+1} = \frac{1}{2}\ln(x^3+1)$ $\quad f'(x) = \left(\frac{1}{2}\right)3x^2\left(\frac{1}{x^3+1}\right)$

d) $f(x) = \sqrt{3x+5}(2x-3)^4$ $\quad \ln f(x) = \frac{1}{2}\ln(3x+5) + 4\ln(2x-3)$ $\quad f'(x) = \left[\left(\frac{3}{2}\right)\left(\frac{1}{3x+5}\right) + 8\left(\frac{1}{2x-3}\right)\right]f(x)$

e) $f(x) = x^{x^2}$ $\quad \ln f(x) = x^2\ln x$ $\quad f'(x) = \left[2x\ln x + x^2\left(\frac{1}{x}\right)\right]f(x) = [1+\ln x]2x^{x^2+1}$

Antiderivatives and the indefinite integral. Including initial value problems.

17. Evaluate the indefinite integral.

a) $\int 1 + x + \sqrt{x}dx = x + \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} + C$

b) $\int \frac{1}{x}dx = \ln|x| + C$

c) $\int \frac{1}{x^2} - x^2 dx = -\frac{1}{x} - \frac{1}{3}x^3 + C$

18. Solve the initial value problem.

a) $f'(x) = 4x + e^x - 2$ and $f(1) = 0$. $f(x) = 2x^2 + e^x - 2x + C$ $\quad 0 = f(1) = 2 + e - 2 + C \Rightarrow C = -e$

$f(x) = 2x^2 + e^x - 2x - e$

b) $f'(x) = x^{\frac{2}{3}}$ and $f(0) = 3$. $f(x) = \frac{3}{5}x^{5/3} + C.$ $\quad 3 = f(0) = C.$ $\quad f(x) = \frac{3}{5}x^{5/3} + 3$

The definite integral and the Fundamental Theorem of Calculus.

19. Evaluate the definite integral.

a) $\int_0^1 1 - x^2 dx = x - \frac{1}{3}x^3 \Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

b) $\int_1^e \frac{1}{x} dx = \ln|x| \Big|_1^e = 1 - 0 = 1$

c) $\int_{\frac{1}{2}}^2 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{\frac{1}{2}}^2 = -\frac{1}{2} + 2 = \frac{3}{2}$

Integration by substitution. For definite and indefinite integrals.

20. Evaluate the integral

a) $\int_0^1 x e^{x^2 - 1} dx$

b) $\int_0^2 \frac{\ln(2x+1)}{2x+1} dx$

c) $\int_e^{e^2} \frac{1}{x \ln x} dx$

Areas under and between curves.

21. Find the area between the curves over the given interval.

a) $y = x^2$ and $y = 2 - x^2$ over $[-1, 1]$

b) $y = x^3$ and $x = y$ over $[-1, 1]$

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6. $x^2 - xy^2 - 2y^4 = 13$

$$2x - (y^2 + 2xyy') - 8y^3y' = 0 \Leftrightarrow (2xy + 8y^3)y' = 2x - y^2$$

$$\Leftrightarrow y' = \frac{2x - y^2}{2xy + 8y^3}$$

7. $V = \pi 16 h$

$$\frac{dV}{dt} = \pi 16 \frac{dh}{dt}$$

$\frac{dh}{dt} = .4 \text{ in/s}$ so $\frac{dV}{dt} = 6.4\pi \frac{\text{in}^3}{\text{sec}}$

8. $(x-1)^2 + y^2 = 5$

$$2(x-1) + 2yy' = 0$$

$$y' = \frac{1-x}{y}$$

At $(0,2)$ the slope is $y' = \frac{1}{2}$

9. $f(x) = \frac{1}{x^2+1}$ $\quad f'(x) = -2x(x^2+1)^{-2}$ $\quad f''(x) = -2(x^2+1)^{-2} + 8x^2(x^2+1)^{-3}$

$$= \frac{-2(x^2+1) + 8x^2}{(x^2+1)^3}$$

$$= \frac{6x^2 - 2}{(x^2+1)^3}$$

$f''(x) = 0 \Leftrightarrow 8x^2 - 2x^2 - 2 = 0$

$\Leftrightarrow 6x^2 = 2$

$\Leftrightarrow x = \pm\sqrt{\frac{1}{3}}$

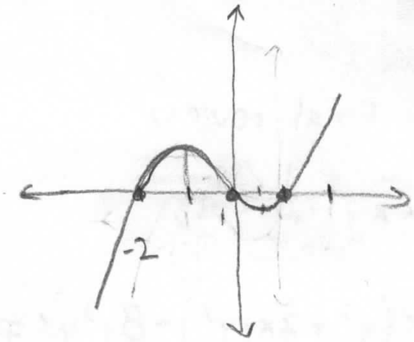
$f''(-1) = \frac{6-2}{2^3} > 0$ $\quad$ concave up on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$

$f''(0) = \frac{-2}{1} < 0$ $\quad$ concave down on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$f''(1) > 0$

$f(x) = x(x^2+x-2) = x(x+2)(x-1)$

10. $f'(x) = 3x^2+2x-2$



$f'(x)=0 \Leftrightarrow 3x^2+2x-2=0$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{4+24}}{6} = \frac{-2 \pm \sqrt{28}}{6}$

$f(-2) = -8+4+4=0$

$f(2) = 8+4-4 = 8$ ← Abs. max

$f\left(\frac{-2-\sqrt{28}}{6}\right) \approx 2.11$ $\quad f\left(\frac{-2+\sqrt{28}}{6}\right) \approx -.63$ ← Abs. min

11. $f'(x) = \frac{x}{x^2+1} = 0 \Leftrightarrow x=0.$

Never undefined.

CP: $x=0$. A min by 1st derivative test.

12. $V = x^2y = 108 \Leftrightarrow y = \frac{108}{x^2}$ (or $x=0$)

Material: $x^2+4xy = x^2+4x\left(\frac{108}{x^2}\right) = x^2+\frac{432}{x}$

$M(x) = x^2 + \frac{432}{x}$

$M'(x) = 2x - \frac{432}{x^2}$ $\quad M'(x)=0 \Leftrightarrow \frac{2x^3-432}{x^2}=0$

$\Leftrightarrow 2x^3-432=0$

$\Leftrightarrow x^3 = 216$

$\Leftrightarrow x = 6$

$M''(x) = 2+\frac{864}{x^3}$ $\quad M''(6)>0$ so $x=6$ is a minimum.

Min material when dimensions are $6 \times 6 \times 3$

14. $P(t) = P_0(1+\frac{r}{n})^{nt}$

$8000 = P(6) = P_0(1+\frac{.06}{12})^{36} \Rightarrow P_0 = 8000(1+\frac{.06}{12})^{-36}$

15. $Q(t) = Q_0 2^{-\frac{t}{1.3}}$ (t in billions of years)

$\frac{Q(t)}{Q_0} = .32 = 2^{-\frac{t}{1.3}} \Rightarrow \ln(.32) = -\frac{t}{1.3}\ln 2$

$\Rightarrow t = \frac{-(1.3)\ln(.32)}{\ln 2} \approx 2.137$

The rock is about 2.137 billion years old.

20. a) $\int_0^1 x e^{x^2-1}\,dx = \int_{-1}^{0} \frac{1}{2} e^u\,du = \frac{1}{2} e^u \Big|_{-1}^{0} = \frac{1}{2}(e^0 - e^{-1}) = \frac{1}{2}\left(1-\frac{1}{e}\right)$

$u = x^2-1$

$\frac{du}{dx} = 2x \quad \frac{1}{2}du = x\,dx$

b) $\int_0^2 \frac{\ln(2x+1)}{2x+1}\,dx = \int_0^{\ln 5} \frac{1}{2}u\,du = \frac{1}{4}u^2\Big|_0^{\ln 5} = \frac{1}{4}(\ln 5)^2$

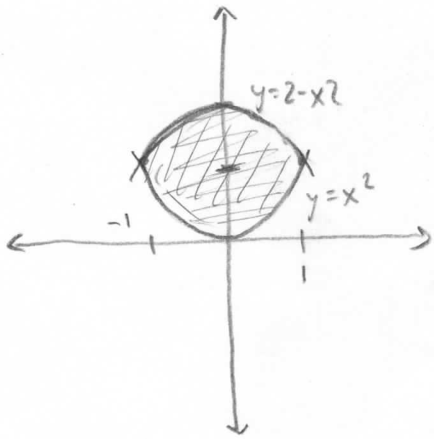
$u = \ln(2x+1)$

$\frac{du}{dx} = \frac{2}{2x+1} \quad \frac{1}{2}du = \frac{1}{2x+1}\,dx$

c) $\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \quad du = \frac{1}{x}dx$



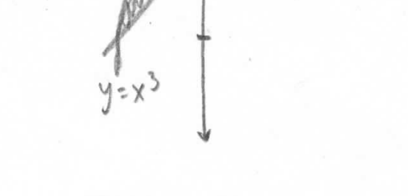
21. a) $y=x^2 \quad y=2-x^2 \quad [-1,1]$

$\int_{-1}^{1} 2-x^2-x^2\,dx = \int_{-1}^{1} 2-2x^2\,dx$

$= 2x - \frac{2}{3}x^3\Big|_{-1}^{1}$

$= 2-\frac{2}{3}-\left(-2+\frac{2}{3}\right) = 4 - \frac{4}{3} = \frac{8}{3}$

b) $y=x^3 \quad y=x \quad [-1,1]$ Area $= \int_{-1}^{0} x^3-x\,dx + \int_0^1 x-x^3\,dx$



$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

$= -\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}$

$= \frac{1}{2}$