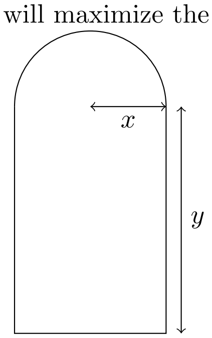
1. a) Find the absolute extrema of $f(x) = x^3 - 6x^2 + 9x$ over the domain $[0, 5]$.
Solution. Find the critical points by solving $f'(x) = 0$. The critical points are $x = 1$ and $x = 3$. Evaluate the function at the critical points and the end points: $f(0) = 0$, $f(1) = 4$, $f(3) = 0$, and $f(5) = 20$. Therefore the absolute minimum value of the function over the interval $[0, 5]$ is 0 (reached at $x = 0$ and at $x = 3$) and the absolute maximum over the interval is 20 (reached at $x = 5$).

b) Find the absolute extrema of $g(x) = 2x^3 - 12x^2 + 18x + 10$ over the domain $[0, 5]$ (try to do this without differentiating).
Solution. Note that $g(x) = 2f(x) + 10$ and thus $g'(x) = 2f'(x)$. This means that g has the same critical points as f. Evaluate $g(x)$ at the critical points and the end points: $g(0) = 10$, $g(1) = 18$, $g(3) = 10$, and $g(5) = 50$. Therefore the absolute minimum value of the function over the interval $[0, 5]$ is 10 (reached at $x = 0$ and at $x = 3$) and the absolute maximum over the interval is 50 (reached at $x = 5$).

2. By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps an open box may be made. If the cardboard is 15 in. long and 8 in. wide, find the dimensions of the box that will yield the maximum volume.
Solution. Let x be the length of the side of one of the squares to be cut away. The volume of the box is then $V(x) = x(15 - 2x)(8 - 2x) = 4x^3 - 46x^2 + 120x$. Note that the domain of x is $[0, 4]$ since only these values make sense for this situation. We're looking for an absolute maximum for $V(x)$ over this domain. Solve $V'(x) = 0$ to find the critical points at $x = \frac{5}{3}$ and $x = 6$. Only $x = \frac{5}{3}$ is in the domain. Evaluate $V(x)$ at $x = 0$, $x = \frac{5}{3}$, and $x = 4$ to find that the absolute maximum volume is $\frac{2450}{27}$, reached when $x = \frac{5}{3}$.

3. A Norman window has the shape of a rectangle surmounted by a semicircle. If a Norman window is to have a perimeter of 28 ft., what dimensions will maximize the area of the window?



Solution. The perimeter of the window is $28 = 2y + x + \pi x$. Solve for y to find

$$y = \frac{28 - (1 + \pi)x}{2}.$$

The area of the window is $A = 2xy + \frac{\pi}{2}x^2$. Substituting in for y gives

$$A(x) = 28x - (1 + \pi)x^2 + \frac{\pi}{2}x^2 = 28x - (1 + \frac{\pi}{2})x^2.$$

Solve $A'(x) = 0$ to find the critical point at $x = \frac{28}{2+\pi}$. Examining the graph of $A(x)$ or using the first or second derivative tests verifies that $A(x)$ reaches a global maximum at $x = \frac{28}{2+\pi}$. The maximum area of the window is therefore $A\left(\frac{28}{2+\pi}\right) = \frac{392}{2+\pi}$.

4. World population is forecast to be

$$P(t) = 0.00074t^3 - 0.0704t^2 + 0.89t + 6.04 \quad (0 \le t \le 4)$$

where t is measured in decades after 2000 and $P(t)$ is measured in billions. When will population peak according to this model?

Solution. Solve $P'(t) = 0$ to find $t \approx 7.12$ and $t \approx 56.3$ (the quadratic equation $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may be helpful). Neither of these points is in our given domain of $[0, 4]$ so we just need to check the original function at the endpoints. We find that $P(0) = 6.04$ and $P(4) \approx 8.52$. Therefore the maximum population over the domain of this model is 8.52 billion, reached at $t = 4$ (the year 2040). Note that if the model could be extended for a couple of decades, then we would find a maximum value at $t \approx 7.12$