HIGHER-ORDER DERIVATIVES

1. On the first worksheet we dealt with a mathematical model for the vertical position of a model rocket. The altitude (in meters) t seconds after launch is given by

$$h(t) = \begin{cases} 40t^2 & \text{if } t \leq 2 \\ 160 + 160(t-2) - 4(t-2)^2 & \text{if } t > 2 \end{cases}$$

This is a piecewise function because the rocket engine stops 2 seconds into the flight, after which the rocket moves only under the influences of gravity and friction.

a) Calculate the derivative $h'(t)$ and interpret what you find. Does the derivative exist at $t = 2$?

For $t<2$: $h'(t) = \frac{d}{dt}\left[40t^2\right] = 80t \ \frac{m}{s}$

For $t>2$: $h'(t) = \frac{d}{dt}\left[160 + 160(t-2) - 4(t-2)^2\right] = 160 - 8(t-2) \ \frac{m}{s}$

Then $\lim_{t \to 2^-} h'(t) = \lim_{t \to 2^-} 80t = 160$ and $\lim_{t \to 2^+} h'(t) = \lim_{t \to 2^+} 160 - 8(t-2) = 160$.

Since these agree, $h'(2) = 160 \frac{m}{s}$. $h'(t)$ is the vertical velocity of the rocket t seconds after launch.

b) Calculate the second derivative $h''(t)$ and interpret what you find. Does the second derivative exist at $t = 2$?

For $t<2$: $h''(t) = \frac{d}{dt}\left[80t\right] = 80 \ \frac{m}{s^2}$

For $t>2$: $h''(t) = \frac{d}{dt}\left[160 - 8(t-2)\right] = -8 \ \frac{m}{s^2}$

$h''(t)$ is the vertical acceleration of the rocket. This time $\lim_{t \to 2^-} h''(t) = 80 \neq -8 = \lim_{t \to 2^+} h''(t)$, so $h''(2)$ DNE. This represents the sudden change from accelerating up to moving only under the influence of gravity (and air resistance).

c) Calculate the third derivative $h^{(3)}(t)$ and interpret what you find.

$h^{(3)}(t) = 0$ except that $h^{(3)}(2)$ DNE (since $h''(2)$ DNE).
This means the rate of change of acceleration is always 0: acceleration is constant (except at the instant the rocket motor stops firing).

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2. Real mathematical models for the absorption and metabolism of a drug are too complex for us right now, but we can work with a simplified example. Suppose the concentration of a intravenously inject drug (in mg/l) t minutes after injection is given by

$$C(t) = \frac{10t}{t^2+1}$$

a) Calculate the derivative $C'(t)$ and interpret what you find. What are the units of $C'(t)$? For which values of t is $C'(t)$ positive? For which values is it negative?

$$C'(t) = \frac{10(t^2+1) - 10t(2t)}{(t^2+1)^2} = \frac{10(1-t^2)}{(t^2+1)^2} \quad \frac{mg/l}{min}$$

$C'(t) > 0$ when $1-t^2>0$: $-1<0<1$.

$C'(t) < 0$ when $1-t^2<0$: $t<-1$ or $t>1$.

→ This means the concentration is increasing

→ the concentration is decreasing.

Note that $t<0$ doesn't really make sense here.

b) Use Desmos to plot the graph $y = C(t)$. Compare the features of the graph with your answers for part a.

Should see a curve with positive slopes for $t<1$ and negative slopes for $t>1$.

c) Calculate the second derivative $C''(t)$. What are the units of $C''(t)$?

$$C''(t) = \frac{d}{dt}\left[\frac{10(1-t^2)}{(t^2+1)^2}\right] = 10\left[\frac{-2t(t^2+1)^2 - (1-t^2)2(t^2+1)(2t)}{(t^2+1)^4}\right] \quad \frac{mg/l}{s^2}$$

When $C''(t)>0$ the concentration is increasing faster (or decreasing slower) as time goes on.

When $C''(t)<0$ the concentration is increasing slower (or decreasing faster) as time goes on.

Challenge. Continue working with $C(t)$ from the last problem. Simplify $C''(t)$ and find the values of t for which $C''(t)$ is positive and the values for which it is negative. Compare with the graph of $y = C(t)$.

$$C(t) = \frac{10t}{t^2+1}$$

$$C'(t) = \frac{10(t^2+1) - 10t(2t)}{(t^2+1)^2} = \frac{10(1-t^2)}{(t^2+1)^2}$$

$$C''(t) = \frac{10\left[-2t(t^2+1)^2 - (1-t^2)4t(t^2+1)\right]}{(t^2+1)^4}$$

$$= \frac{10\left[-2t(t^2+1) - (1-t^2)4t\right]}{(t^2+1)^3}$$

$$= \frac{10\left[-2t^3-2t-4t+4t^3\right]}{(t^2+1)^3}$$

$$= \frac{10(2t^3-6t)}{(t^2+1)^3} = \frac{20t(t^2-3)}{(t^2+1)^3}$$

$= 0$ when $t=0$ or $t = \pm\sqrt{3}$

$C''(t) > 0$ when $t > \sqrt{3}$ (or $-\sqrt{3} < t < 0$)

$C''(t) < 0$ when $0 < t < \sqrt{3}$ (or $t < -\sqrt{3}$)