1. Radioactive carbon-14 (^{14}C) is constantly being formed in the atmosphere when cosmic rays hit nitrogen. This ^{14}C is absorbed by plants (in the form of carbon dioxide), so living plants have the same proportion of ^{14}C to regular ^{12}C as the atmosphere. After the plant dies, however, the radioactive decay of ^{14}C means that this proportion diminishes over time. The rate of decay is given by the half-life of ^{14}C: the proportion of ^{14}C to ^{12}C is reduced by half every 5730 years. In other words, the rate of change of this quantity is determined by the quantity itself. If we let $P(t)$ be the proportion of ^{14}C to ^{12}C t years after death, the we have $P'(t) = kP(t)$ for some constant k. This can be solved to find

$$\boxed{P(t) = P(0)\,(0.5)^{\frac{t}{5370}}} \rightarrow \frac{t}{5730} \quad \text{typo}$$

If ~~we~~ $P(0)$ is the same now as it was in the (geologically) recent past, we can use this to estimate the age of dead plants using radiocarbon dating.

a) One important use of radiocarbon dating found that the oldest plants in the northern US are about 13,000 years old (this is when the glaciers melted). What percent of ^{14}C remained in these plants? That is, find the value of $P(13000)/P(0)$.

$$\frac{P(13000)}{P(0)} = (0.5)^{\frac{13000}{5730}} \approx 0.2075 \quad \text{so} \quad \boxed{20.75\% \text{ remained}}$$

(18.67% of the original ^{14}C remained with the typo of 5370)

b) Corn was domesticated in Central America and later brought to North America. Suppose that measurements of the oldest corn samples from the US have $P(t)/P(0) = 0.60$. How old is this corn?

$$0.6 = \frac{P(t)}{P(0)} = (0.5)^{\frac{t}{5730}} \Rightarrow \ln(0.6) = \frac{t}{5730}\ln(0.5)$$

$$\Rightarrow t = 5730\,\frac{\ln(0.6)}{\ln(0.5)} \approx \boxed{4222.8 \text{ yrs old}}$$

(3957.5 yrs old with the typo)

c) Find the value of k such that $\frac{P'(t)}{\cancel{P(t)}} = kP(t)$.

$$P'(t) = \frac{d}{dt}\left[P(0)\,e^{-\frac{\ln 2}{5730}t}\right] = P(0)\left(-\frac{\ln 2}{5730}\right)e^{-\frac{\ln 2}{5730}t}$$

$$= -\frac{\ln 2}{5730}\,P(t)$$

Date: October 5, 2022.

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $T'(t) = -k(T(t) - T_m)$ where k is a positive constant of proportionality, T_m is the (constant) temperature of the environment, and $T(t)$ is the temperature of object as a function of time. This can be solved to find

$$\boxed{T(t) = T_m + Ce^{-kt}}$$

(where C is a constant). Use Newton's law of cooling to solve the next two problems.

2 (CSI MATH 148). A body is found outside on a 5°C day. At 12:15 its temperature is 35°C and at 12:45 its temperature is 33°C.

a) Use the two points to find a formula for $T(t)$, the temperature of the body t hours after 12:15. Note that this means $T(0) = 35$.

$$35 = T(0) = 5 + Ce^{-k0} = 5 + C \Rightarrow C = 30$$

$$33 = T(0.5) = 5 + 30e^{-k/2} \Rightarrow \frac{28}{30} = e^{-k/2}$$

$$\Rightarrow k = 2\ln\left(\frac{30}{28}\right) \approx 0.1379857$$

$$T(t) = 5 + 30e^{2\ln\left(\frac{28}{30}\right)t} = 5 + 30\left(\frac{28}{30}\right)^{2t}$$

b) Human body temperature is 37°C. Use this to find the time of death.

$$37 = T(t) = 5 + 30\left(\frac{28}{30}\right)^{2t}$$

$$\frac{32}{30} = \left(\frac{28}{30}\right)^{2t}$$

$$\frac{\ln(32/30)}{2\ln(28/30)} = t$$

$t \approx -0.46771876 \approx -28$ minutes

time of death: 11:47

Challenge. A cup of boiling water (212° F) is placed outside. One minute later the temperature of the water is 152° F. After another minute the temperature is 112° F. What is the outside temperature?

Solution to the challenge problem.

$T(0) = 212$

$T(1) = 152$

$T(2) = 112$

and $T(t) = T_m + Ce^{-kt}$

$212 = T_m + Ce^{-k \cdot 0} = T_m + C \Rightarrow T_m = 212 - C$

$152 = T(1) = T_m + Ce^{-k} = 212 - C + Ce^{-k} = 212 - C(1-e^{-k})$

$112 = T(2) = T_m + Ce^{-2k} = 212 - C + Ce^{-2k} = 212 - C(1-e^{-2k})$

$60 = C(1-e^{-k})$ sub in

$100 = C(1-e^{-2k})$

$40 + C(1-e^{-k}) = C(1-e^{-2k}) \Rightarrow 40 = Ce^{-k}(1-e^{-k})$

$= e^{-k}(60)$

$\Rightarrow e^{-k} = \frac{40}{60} = \frac{2}{3}$

$60 = C(1-\frac{2}{3}) = C(\frac{1}{3}) \Rightarrow C = 180$

$\Rightarrow T_m = 212 - C = 212 - 180 = \boxed{32}$