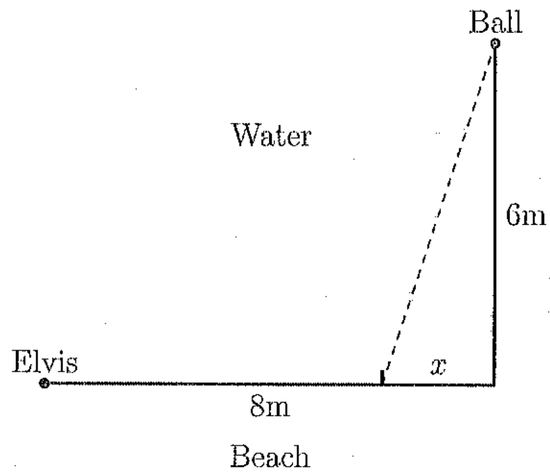
FIRST DERIVATIVES AND OPTIMIZATION

1. A dog named Elvis is on the edge of a lake and his ball is in the water 8 meters down the shore and 6 meters into the water. The diagram shows the view from above. Elvis can run along the beach at a speed of 3 m/s and he can swim at 1 m/s. Elvis wants to get the ball as quickly as possible.



a) How long does it take Elvis to get to the ball if he swims as little as possible (so he runs 8m down the beach then swims 6m out to the ball)?

Run time $8\,m / 3\frac{m}{s} = \frac{8}{3}$ s

Swim time $6\,m / 1\frac{m}{s} = 6$ s

Add: total time $\frac{8}{3} + 6 = \frac{26}{3}$ s

b) How long does it take Elvis to get to the ball if he swims all the way (diagonally straight to the ball)?

Distance: $\sqrt{8^2 + 6^2} = 10$ m

time $10\,m / 1\frac{m}{s} = 10$ s.

c) Now find the shortest possible time to the ball.

i) Find a function $f(x)$ for the time it takes Elvis to get to the ball if he runs down the beach to a distance x from the point on the shore closest to the ball and then swims.

ii) Find the **critical values** of f: differentiate this function and solve $f'(x) = 0$ for x.

iii) Verify that the value you found for x gives the least time to the ball.

Run distance: $8 - x$ m. Run time: $\frac{8-x}{3}$ s

Swim distance: $\sqrt{x^2+36}$. Swim time $\sqrt{x^2+36}$ s

Add: $\boxed{f(x) = \frac{8-x}{3} + \sqrt{x^2+36}}$

$f'(x) = -\frac{1}{3} + \frac{x}{\sqrt{x^2+36}}$ ← defined everywhere.

$0 = -\frac{1}{3} + \frac{x}{\sqrt{x^2+36}} \Rightarrow \frac{1}{3} = \frac{x}{\sqrt{x^2+36}} \Rightarrow \sqrt{x^2+36} = 3x \Rightarrow x^2 + 36 = 9x^2$

$\Rightarrow x = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}}$.

$x = -\frac{3}{\sqrt{2}}$ can't be fastest. Check $x = \frac{3}{\sqrt{2}}$ $\quad \overset{-}{\longleftarrow} \underset{3/\sqrt{2}}{|} \overset{+}{\longrightarrow} \; f'(x)$.

Min @ $x = \frac{3}{\sqrt{2}}$. Min time: $f\left(\frac{3}{\sqrt{2}}\right) = \frac{8 - \frac{3}{\sqrt{2}}}{3} + \sqrt{\frac{9}{2} + 36} = \frac{8}{3} + \frac{-1+9}{\sqrt{2}} = \frac{8}{3} + \frac{8}{\sqrt{2}}$ ≈ 8.32 s

Challenge. Biologists have determined that if a fish swims at a speed v through the water, then its energy expenditure is proportional to v^3. Suppose that a hypothetical fish swims a distance of L meters against a fixed current of u meters per second (think of salmon swimming upstream). The energy expended by the fish is then

$$E(v) = av^3\left(\frac{L}{v-u}\right)$$

where a is a constant determined by the size and shape of the fish. What speed minimizes E? (Your answer will depend on the water speed u).

$$E'(v) = aL\left[\frac{3v^2(v-u)-v^3}{(v-u)^2}\right] = aL\left[\frac{2v^3-3v^2u}{(v-u)^2}\right] = aLv^2\left[\frac{2v-3u}{(v-u)^2}\right]$$

Undefined when $v=u$ & this would mean the fish doesn't move across the ground at all; 0 distance swum.

$$0 = aLv^2\left[\frac{2v-3u}{(v-u)^2}\right] \Rightarrow 0 = v^2(2v-3u) \Rightarrow v=0 \text{ or } v=\frac{3}{2}u.$$

$-\ 0\ -\ u\ -\ 0\ +$

$0 \quad u \quad \frac{3}{2}u$

Min

$$E'(v) = \underbrace{aLv^2}_{+ \text{ always}}\left[\frac{2v-3u}{(v-u)^2}\right]$$

determines the sign of $E'(v)$