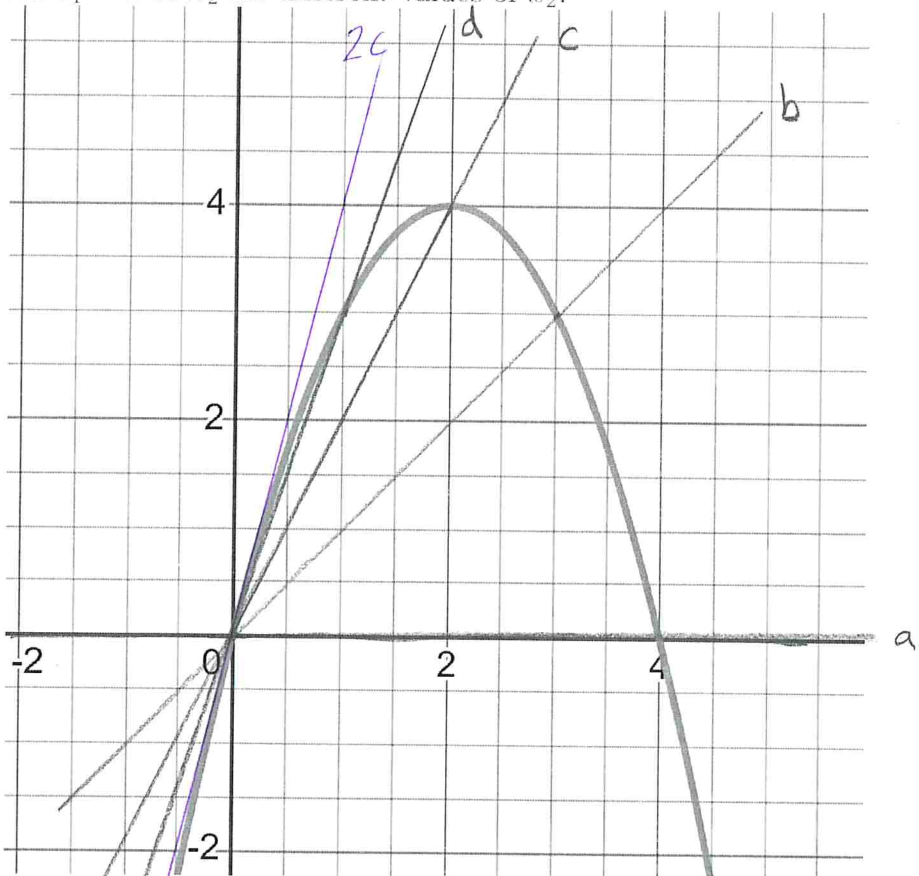
RATES OF CHANGE AND DERIVATIVES

Definition. The **average rate of change** of $f(x)$ as x changes from x_1 to x_2 (when $x_1 \neq x_2$) is

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

This is the same as the **slope of the secant line** through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

1. Shown below is a graph of the function $f(x) = 4x - x^2$. We will consider the average rate of change from $x_1 = 0$ to x_2 for different values of x_2.



a) Sketch the secant line from $(0, f(0))$ to $(4, f(4))$ and calculate its slope.

secant line is the x-axis, Slope 0.

$f(0)=0$

b) Sketch the secant line from $(0, f(0))$ to $(3, f(3))$ and calculate its slope.

$f(3) = 4(3) - 3^2 = 12 - 9 = 3$. Slope $\frac{3-0}{3-0} = 1$.

c) Sketch the secant line from $(0, f(0))$ to $(2, f(2))$ and calculate its slope.

$f(2) = 4(2) - 2^2 = 8 - 4 = 4$. Slope $\frac{4-0}{2-0} = 2$.

d) Sketch the secant line from $(0, f(0))$ to $(1, f(1))$ and calculate its slope.

$f(1) = 4(1) - 1^2 = 3$. Slope $\frac{3-0}{1-0} = 3$.

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Definition. The **instantaneous rate of change** (also called the **derivative**) of $f(x)$ at $x = a$ is

$$\boxed{f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

This is the same as the **slope of the tangent line** at the point $(a, f(a))$.

2. Continue working with the function $f(x) = 4x - x^2$. Our goal now is to find the instantaneous rate of change of $f'(0)$.

a) Does your work for problem 1 suggest an answer? Make a guess if you can.

$0, 1, 2, 3, \dots$ Maybe 4?

b) Evaluate a limit to find the exact value of $f'(0)$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{4x - x^2}{x} = \lim_{x \to 0} 4 - x = 4.$$

c) Find an equation for the tangent line at the point $(0, 0)$ (use the slope you just calculated). Add this line to the graph on page 1.

$y = 4x$. Sketched in purple.

Definition. The **derivative** of $f(x)$ is $\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$

Challenge (skip this and come back if you have time). Calculate $f'(x)$ for the following functions.

a) $f(x) = 4x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4(x+h) - 4x}{h} = \lim_{h \to 0} \frac{4h}{h} = 4$$

b) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh - h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x - h)}{h} = \lim_{h \to 0} 2x - h = 2x$$

c) $f(x) = 4x - x^2$

$$f'(x) = \lim_{h \to 0} \frac{4(x+h) - (x+h)^2 - [4x - x^2]}{h}$$

$$= \lim_{h \to 0} \left[\frac{4(x+h) - 4x}{h} - \frac{(x+h)^2 - x^2}{h} \right]$$

Already evaluated both limits in parts a & b.

$$= 4 - 2x$$

3. The microscope equation says that for values of x near a

$$f(x) \approx f'(a)(x-a) + f(a).$$

Note that this just says that the tangent line $y = f'(a)(x-a) + f(a)$ must be close to the graph $y = f(x)$ near the point $(a, f(a))$. Your goal here is to use the microscope equation for the function $f(x) = x^{1/3}$ to estimate $1.12^{1/3}$ **without using a calculator**.

a) First we need a number a near 1.12 at which we can easily evaluate $f(a)$. What should we use?

$a=1 \qquad f(1) = 1^{1/3} = 1$

b) Now we need $f'(a)$. Use the fact that $f'(x) = \frac{1}{3}x^{-2/3}$.

$f'(1) = \left(\frac{1}{3}\right) 1^{-2/3} = \frac{1}{3}$

c) Find the microscope equation.

$f(x) \approx \frac{1}{3}(x-1) + 1$

d) Use the microscope equation to estimate $1.12^{1/3}$ and compare your estimate with the more exact estimate of 1.0384988.

$1.12^{1/3} = f(1.12) \approx \frac{1}{3}(1.12-1) + 1 = \frac{0.12}{3} + 1 = 0.04 + 1 = 1.04.$

Pretty close to the more exact answer of 1.0384988

Bonus: $1.09^{1/3} \approx \frac{1}{3}(1.09-1) + 1 = 1.03$

More exact: $1.09^{1/3} \approx 1.0291424657$

Note: the only way to write the exact number here is as $1.12^{1/3}$ or $\sqrt[3]{1.12}$. Any decimal you write (like 1.0384988) will be an approximation.

Question: how do you think calculators know that $1.12^{1/3} \approx 1.03849882037$?