Solutions

THE PRODUCT, QUOTIENT, AND CHAIN RULES

Theorem. Building blocks:

i. $\frac{d}{dx}[c] = 0$ for any constant c

ii. $\frac{d}{dx}[x^n] = nx^{n-1}$ $(n \neq 0)$

iii. $\frac{d}{dx}[e^x] = e^x$

iv. $\frac{d}{dx}[\sin x] = \cos x$

v. $\frac{d}{dx}[\cos x] = -\sin x$

Product rule: $\boxed{(fg)' = f'g + g'f}$

Quotient rule: $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{[g]^2}}$

Chain rule: $\boxed{(f(g(x)))' = f'(g(x))\,g'(x)}$

1. Observe that $\frac{1}{x} - 9 = \frac{1-9x}{x} = \left(\frac{1}{\sqrt{x}} - 3\right)\left(\frac{1}{\sqrt{x}} + 3\right)$. Use this to differentiate $f(x) = \frac{1}{x} - 9$ in three different ways.

a) Differentiate directly: $\frac{d}{dx}\left[\frac{1}{x} - 9\right] = \frac{d}{dx}\left[x^{-1} - 9\right] = -x^{-2} = -\frac{1}{x^2}$

the derivative is done here

b) Use the quotient rule: $\frac{d}{dx}\left[\frac{1-9x}{x}\right] = \frac{-9x - 1(1-9x)}{x^2} = \frac{-9x-(1-9x)}{x^2}$

$= -\frac{1}{x^2}$ ✓

$f(x) = 1-9x \qquad f'(x) = -9$

$g(x) = x \qquad g'(x) = 1$

c) Use the product rule: $\frac{d}{dx}\left[\left(\frac{1}{\sqrt{x}} - 3\right)\left(\frac{1}{\sqrt{x}} + 3\right)\right] = -\frac{1}{2}x^{-3/2}\left(x^{-\frac{1}{2}} + 3\right) + \left(-\frac{1}{2}x^{-3/2}\right)\left(x^{-\frac{1}{2}} - 3\right)$

$f(x) = \frac{1}{\sqrt{x}} - 3 = x^{-\frac{1}{2}} - 3 \qquad f'(x) = -\frac{1}{2}x^{-3/2}$

$g(x) = \frac{1}{\sqrt{x}} + 3 = x^{-\frac{1}{2}} + 3 \qquad g'(x) = -\frac{1}{2}x^{-3/2}$

$= -\frac{1}{2}x^{-3/2}\left[x^{-\frac{1}{2}} + 3 + (x^{-\frac{1}{2}} - 3)\right]$

$= -\frac{1}{2}x^{-3/2}\left[2x^{-1/2}\right]$

$= -x^{-3/2 - 1/2} = -x^{-2} = -\frac{1}{x^2}$ ✓

d) All three of your solutions should be equal. Are they? Yes, but it takes some algebra to verify this.

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2. Find an equation for the line tangent to the graph $y = \sqrt{x^2 + 3x}$ at the point $(1, 2)$.

$$\frac{d}{dx}\left[(x^2+3x)^{1/2}\right] = \frac{1}{2}(x^2+3x)^{-1/2}\frac{d}{dx}(x^2+3x)$$

$$= \frac{1}{2}(x^2+3x)^{-1/2}(2x+3)$$

$$@\, x=1: \quad y' = \frac{1}{2}(1+3)^{-1/2}(2+3) = \frac{1}{2}\left(\frac{1}{2}\right)(5) = \frac{5}{4}$$

Tangent line: $y - 2 = \frac{5}{4}(x-1)$

3. Use the chain rule (and any other appropriate rules) to find the following derivatives:

a) $\frac{d}{dx}[(x^2-1)^5] = 5(x^2-1)^4\frac{d}{dx}(x^2-1) = 5(x^2-1)^4(2x) = 10x(x^2-1)^4$

b) $\frac{d}{dx}\left[\frac{1}{\sqrt{7x+3}}\right] = \frac{d}{dx}\left[(7x+3)^{-\frac{1}{2}}\right] = -\frac{1}{2}(7x+3)^{-\frac{3}{2}}\frac{d}{dx}(7x+3)$

$$= -\frac{1}{2}(7x+3)^{-3/2}(7) = -\frac{7}{2}(7x+3)^{-3/2}$$

c) $\frac{d}{dx}\left[e^{x^2}\right] = e^{(x^2)}\frac{d}{dx}(x^2) = e^{(x^2)}(2x)$

d) $\frac{d}{dx}\left[\sin\left(\frac{e^x}{x}\right)\right] = \cos\left(\frac{e^x}{x}\right)\frac{d}{dx}\left(\frac{e^x}{x}\right) = \cos\left(\frac{e^x}{x}\right)\left[\frac{xe^x - e^x}{x^2}\right]$