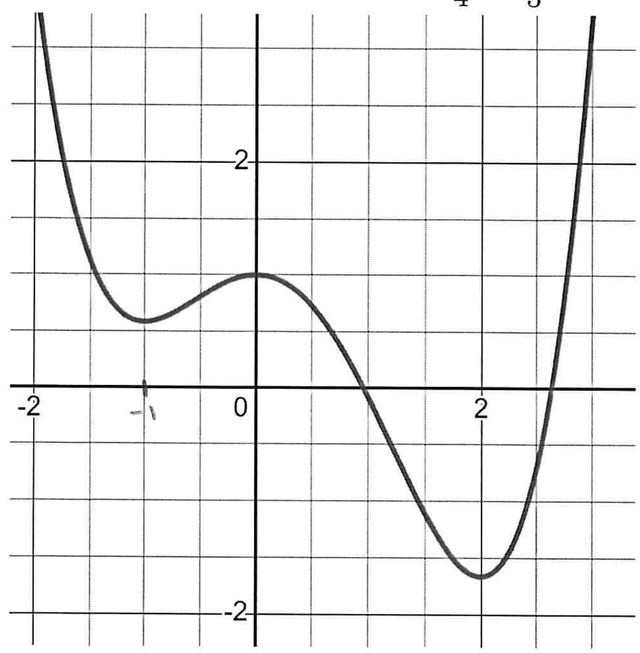
FIRST DERIVATIVES AND OPTIMIZATION Solutions

1. The peaks and valleys we've been seeing in graphs are more formally called **local maxima** and **local minima**, respectively. The goal of this problem is to explore the relationship between these local extremes and the sign of the derivative. Shown below is the graph $y = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + 1$.



a) Use the graph to estimate the x-coordinates of the one local maximum and two local minima.

Local min @ $x=-1$ and $x=2$. Local max @ $x=0$.

b) Differentiate $y = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + 1$ to find y'.

$y' = x^3 - x^2 - 2x$

c) Solve $y' = 0$ for x. You should get three solutions.

$0 = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x-2)(x+1)$

$\Rightarrow x=-1,\ x=0,$ or $x=2$ (same x values as part a)

d) Make a sign chart for y' (use the graph above to help).

$-\quad 0 \quad +\quad 0 \quad -\quad 0 \quad +\quad y'$

$-1 \quad 0 \quad 2 \quad x$

e) How do the signs of y' correspond to the local extremes?

$-\ 0\ +$ $\Rightarrow$ local min

$+\ 0\ -$ $\Rightarrow$ local max

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2. The goal of this problem is to find the local extremes of $f(x) = \frac{x^3}{3} - 2x^2 + 3x - 1$ **without having to draw a graph.**

a) Differentiate $f(x) = \frac{x^3}{3} - 2x^2 + 3x - 1$ to find $f'(x)$.

$f'(x) = x^2 - 4x + 3$

b) Solve $f'(x) = 0$. The x-values you find are called **critical values** of the function f.

$0 = x^2 - 4x + 3 = (x-3)(x-1)$

$\Rightarrow x = 3$ or $x = 1$

c) Make a sign chart for $f'(x)$. Hint: the critical values divide the number line into intervals. Evaluating $f'(x)$ at one point in each interval tells you the sign for the whole interval.

$+ \quad 0 \quad - \quad 0 \quad + \quad f'(x)$

$1 \qquad 3 \qquad x$

$f'(0) = 3 > 0$

$f'(2) = 4 - 8 + 3 = -1 < 0$

$f'(4) = 16 - 16 + 3 = 3 > 0$

d) Interpret your sign chart (according to what you found in 1e) to locate the local maximum and the local minimum of $f(x)$.

Local max @ $x = 1$

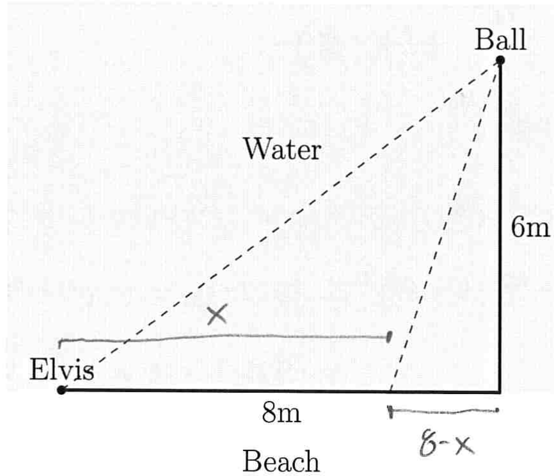
Local min @ $x = 3$

e) Plug the critical values into $f(x)$ to find the y-coordinates of the local maximum and local minimum.

$f(1) = \frac{1}{3} - 2 + 3 - 1 = \frac{1}{3}$ Local max of $\frac{1}{3}$ at $x = 1$

$f(3) = 9 - 18 + 9 - 1 = -1$ Local min of -1 at $x = 3$

3. A dog named Elvis (who happens to be very good at calculus) is on the edge of a lake and his ball is in the water 8 meters down the shore and 6 meters into the water. The diagram below shows an overhead view. Elvis can run along the beach at a speed of 3 m/s and he can swim at 1 m/s. Elvis wants to get the ball as quickly as possible.



a) First calculate how long it takes Elvis to get to the ball if he swims all the way (diagonally straight to the ball)? Hint: figure out how far he needs to swim using the Pythagorean theorem ($a^2+b^2=c^2$), then calculate how long it takes to cover that distance when he swims at 1 m/s.

Distance: $\sqrt{8^2+6^2} = 10$ m. 10 m @ 1 m/s takes $\frac{10\text{ m}}{1\text{ m/s}} = 10$ s

b) Now calculate how long it takes Elvis to get to the ball if he swims as little as possible. That means he runs 8m down the beach then swims 6m out to the ball.

8 m @ 3 m/s takes 8/3 s
6 m @ 1 m/s takes 6/1 = 6 s

add to get total time: $\frac{8}{3}+6 = \frac{26}{3} \approx 8.67$ s.

c) Elvis consistently finds a faster route to the ball by running part way down the beach, then swimming a short diagonal to the ball. Your goal is to find the shortest possible time to the ball.

i) Find a function $f(x)$ for the time it takes Elvis to get to the ball if he runs down the beach x meters, then swims straight to the ball. Hint: use the Pythagorean theorem to find the swimming distance as a function of x.

ii) Find the **critical values** of f: differentiate this function and solve $f'(x)=0$ for x.

iii) Verify that the value you found for x gives the least time to the ball by making a sign chart for $f'(x)$.

i) $f(x) = \frac{x}{3} + \sqrt{6^2+(8-x)^2} = \frac{x}{3} + \sqrt{100-16x+x^2}$

ii) $f'(x) = \frac{1}{3} + \frac{1}{2}\left(100-16x+x^2\right)^{-\frac{1}{2}}(-16+2x)$ chain rule

$= \dfrac{\left(100-16x+x^2\right)^{\frac{1}{2}} + 3(x-8)}{3\left(100-16x+x^2\right)^{1/2}}$ algebra

May also use (possibly easier)

$f'(x) = \frac{1}{3} + \frac{1}{2}\left[6+(8-x)^2\right]^{-\frac{1}{2}}\left[2(8-x)(-1)\right]$ double chain rule

$= \frac{1}{3} + \left[6+(8-x)^2\right]^{-\frac{1}{2}}(x-8)$ algebra

$$0 = f'(x) = \frac{(100-16x+x^2)^{1/2} + 3(x-8)}{3(100-16x+x^2)^{1/2}}$$ only when the numerator is 0.

$$\Rightarrow 0 = (100-16x+x^2)^{1/2} + 3(x-8)$$

$$\Rightarrow -3(x-8) = (100-16x+x^2)^{1/2}$$

$$\Rightarrow 9(x-8)^2 = 100-16x+x^2$$

$$\Rightarrow 0 = 100-16x+x^2-9(x-8)^2 = 100-16x+x^2-9(x^2-16x+64)$$

$$= -476+128x-8x^2$$

$$= -4(119-32x+2x^2)$$

$$\Rightarrow x = \frac{32 \pm \sqrt{1024-952}}{4} = 8 \pm \frac{3}{\sqrt{2}}$$

$$8 - \frac{3}{\sqrt{2}} \approx 5.8786796564\text{4}$$

$$8 + \frac{3}{\sqrt{2}} \approx 10.1213203436$$

iii)

$-$	0	$+$		0	$+$	$f'(x)$
	$8-\frac{3}{\sqrt{2}}$		8	$8+\frac{3}{\sqrt{2}}$		

$$f'(0) = \frac{10-24}{30} < 0$$

$$f'(8) = \frac{6+0}{18} > 0$$

We know $x > 8$ doesn't make sense as a way to minimize time to the ball.

Min must be @ $x = 8 - \frac{3}{\sqrt{2}}$.

the min time to the ball is $f\left(8-\frac{3}{\sqrt{2}}\right) = \frac{8-3/\sqrt{2}}{3} + \sqrt{36+\left(8-\left(8-\frac{3}{\sqrt{2}}\right)\right)^2}$

$$= \frac{8}{3} - \frac{1}{\sqrt{2}} + \sqrt{36+\frac{9}{2}}$$

$$\approx 8.323521 \text{ seconds}$$

Challenge. Biologists have determined that if a fish swims at a speed v through the water, then its energy expenditure is proportional to v^3. Suppose that a hypothetical fish swims a distance of L meters against a fixed current of u meters per second (think of salmon swimming upstream). The energy expended by the fish is then

$$E(v) = av^3\left(\frac{L}{v-u}\right)$$

where a is a constant determined by the size and shape of the fish. What speed minimizes E? (Your answer will depend on the water speed u).