Solutions

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. The amount of money in an account earning interest with an APR of 8% per year depends on how often the interested in compounded. Let A_0 be the initial amount invested. If interest is compounded n times per year, then the amount after t years is

$$A(t) = A_0\left(1 + \frac{0.08}{n}\right)^{nt}$$

If interest is compounded continuously, then the amount after t years is

$$A(t) = A_0 e^{0.08t}$$

How long does it take the amount to reach 10 times its original value if interest is compounded...

a) ...once per year?

$$10 = \left(1 + \frac{0.08}{1}\right)^t = (1.08)^t$$

$$\ln 10 = t \ln(1.08)$$

$$t = \frac{\ln 10}{\ln 1.08} \approx 29.92 \text{ yrs}$$

b) ...once per month?

$$10 = \left(1 + \frac{0.08}{12}\right)^{12t}$$

$$\ln 10 = 12t \ln\left(1 + \frac{0.08}{12}\right)$$

$$t = \frac{\ln 10}{12 \ln\left(1 + \frac{0.08}{12}\right)} \approx 28.88 \text{ yrs}$$

c) ...continuously?

$$10 = e^{0.08t}$$

$$\ln 10 = 0.08t$$

$$t = \frac{\ln 10}{0.08} \approx 28.78 \text{ yrs}$$

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2. Radioactive carbon-14 is constantly being formed in the atmosphere when cosmic rays hit nitrogen. Carbon is absorbed by plants in the form of carbon dioxide, so living plants have the same proportion of carbon-14 to regular carbon-12 as the atmosphere. After the plant dies, however, the radioactive decay of carbon means that this proportion diminishes over time. The rate of decay is given by the half-life: the amount of carbon-14 is reduced by half every 5730 years. If we let $P(t)$ be the proportion of carbon-14 to carbon-12 t years after death, the we have $P' = rP$ for some constant r. This means that $\frac{P(t)}{P(0)} = e^{rt}$ ~~for some constant r.~~

a) Use the fact that $\dfrac{P(5730)}{P(0)} = 0.5$ to find r.

$$0.5 = e^{r(5730)}$$

$$r = \frac{\ln 0.5}{5730} \approx -0.00012096809$$

b) Radiocarbon dating uses measurements of the ratio of carbon-14 to carbon-12 to estimate the age of organic remains. One important use of radiocarbon dating found that the oldest plants in the northern US are about 13,000 years old, which tells us when the glaciers melted most recently. What percent of carbon-14 remained in these plants? That is, find the value of $\frac{P(13000)}{P(0)}$.

$$\frac{P(13000)}{P(0)} = e^{\left(\frac{\ln 0.5}{5730}\right)13000} \approx 0.2075$$

20.75% remained

c) Corn was domesticated in Central America and later brought to North America. Measurements of some of the oldest corn samples from the US have $P(t)/P(0) \approx 0.60$. How old is this corn?

$$0.6 = e^{\left(\frac{\ln 0.5}{5730}\right)t}$$

$$\ln 0.6 = t\left(\frac{\ln 0.5}{5730}\right)$$

$$t = \left(\frac{\ln 0.6}{\ln 0.5}\right)5730 \approx 4222.81 \text{ yrs}$$

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $y' = -k(y - T_e)$ where k is a positive constant, T_e is the temperature of the environment (which we assume is constant), and y is the temperature of object as a function of time. This can be solved to find

$$\boxed{y = T_e + Ce^{-kt}}$$

(where C is a new constant). Use Newton's law of cooling to solve the next two problems.

3 (CSI MATH 148). A body is found outside on a 5°C day. At 12:15 its temperature is 35°C and at 12:45 its temperature is 33°C.

a) Use this information to find the values of the constants in $y = T_e + Ce^{-kt}$. Suggestion: measure time starting at 12:15 so $y(0) = 35$.

$T_e = 5$

$y(0) = 35 = 5 + Ce^{-k\cdot 0} = 5 + C \Rightarrow C = 30$

$y(30) = 33 = 5 + 30e^{-k\cdot 30}$ OR $y(0.5) = 33 = 5 + 30e^{-k\cdot 0.5}$

$\frac{28}{30} = e^{-k\cdot 30}$ $\qquad$ $\frac{28}{30} = e^{-k\cdot 0.5}$

$k = \frac{\ln(\frac{28}{30})}{-30} \approx 0.00229976238$ $\qquad$ $k = -2\ln(\frac{28}{30}) \approx 0.13798574$

$y(t) = 5 + 30e^{\frac{\ln(\frac{28}{30})}{30}t}$ $\qquad$ $y(t) = 5 + 30e^{2\ln(\frac{28}{30})t}$

t measured in minutes $\qquad$ t measured in hours

b) Human body temperature is 37°C. Use this to find the time of death.

$37 = 5 + 30e^{\frac{\ln(\frac{28}{30})}{30}t}$

$\frac{32}{30} = e^{\frac{\ln(\frac{28}{30})}{30}t}$

$t = \frac{30\ln(\frac{32}{30})}{\ln(\frac{28}{30})} \approx -28.06$ minutes

Time of death: 28 minutes before 12:15: about 11:47

Challenge. A cup of boiling water (212° F) is placed outside. One minute later the temperature of the water is 152° F. After another minute the temperature is 112° F. What is the outside temperature?

Challenge solution:

$$y(0) = 212 = T + Ce^{-k \cdot 0} \quad \text{so} \quad \boxed{\begin{aligned} 212 &= T + C \\ 152 &= T + Ce^{-k} \\ 112 &= T + Ce^{-2k} \end{aligned}}$$

$$y(1) = 152 = T + Ce^{-k}$$

$$y(2) = 112 = T + Ce^{-2k}$$

solve for T

$$212 = T + C \Rightarrow C = 212 - T$$

$$\text{so } 152 = T + (212 - T)e^{-k} \Rightarrow e^{-k} = \frac{152 - T}{212 - T}$$

$$\text{and } 112 = T + (212 - T)e^{-2k} \quad \text{so} \quad 112 = T + (212 - T)\left[\frac{152 - T}{212 - T}\right]^2$$

$$= T + \frac{(152 - T)^2}{212 - T}$$

$$\Rightarrow (112 - T)(212 - T) = (152 - T)^2$$

$$\Rightarrow 23744 - 324T + \cancel{T^2} = 23104 - 304T + \cancel{T^2}$$

$$\Rightarrow 20T = -640$$

$$\Rightarrow T = -32^\circ F$$