AVERAGE AND INSTANTANEOUS VELOCITY

The altitude of a model rocket (in meters) t seconds after launch is given by

$$f(t) = \begin{cases} 40t^2 & \text{if } t \leq 2 \\ 160 + 160(t-2) - 4(t-2)^2 & \text{if } t > 2 \end{cases}$$

This is a piecewise function because the rocket engine stops 2 seconds into the flight, after which the rocket moves only under the influences of gravity and friction.

1. Calculate $f(0)$, $f(1)$, $f(1.5)$, and $f(2)$ (you'll need these for the following problems).

$f(0) = 40(0)^2 = 0$ $\qquad$ $f(2) = 40(2)^2 = 160$

$f(1) = 40(1)^2 = 40$

$f(1.5) = f(\frac{3}{2}) \cdot 40(\frac{3}{2})^2 = 90$

Definition. The **average velocity** of the rocket between time t_1 and t_2 is

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

2. Calculate the average velocity of the rocket for ...

a) ... the first 2 seconds of the rocket's flight ($t = 0$ to $t = 2$).

$$\frac{f(2) - f(0)}{2-0} = \frac{160-0}{2} = 80 \frac{m}{s}$$

b) ... $t = 1$ and $t = 2$.

$$\frac{f(2) - f(1)}{2-1} = \frac{160-40}{1} = 120 \frac{m}{s}$$

c) ... $t = 1$ and $t = 1.5$.

$$\frac{f(1.5) - f(1)}{1.5-1} = \frac{90-40}{0.5} = 100 \frac{m}{s}$$

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Definition. The **instantaneous velocity** of the rocket at time t_1 is the limit of the average velocity as t_2 approaches t_1: $\lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}$. This often gets re-expressed using $h = t_2 - t_1$: $\boxed{\lim_{h \to 0} \frac{f(t_1 + h) - f(t_1)}{h}}$.
We'll look more at limits in the next section (tomorrow), but try to make sense of things as best you can now. Note: this gives the *instantaneous rate of change* of whatever quantity is given by the function.

3. Use the following table (fill in the blanks in the third column) to estimate the instantaneous velocity of the rocket at time $t = 1$. Call this velocity $v(1)$.

h	$f(1+h)$	average velocity from $t = 1$ to $t = 1 + h$
0.5	90	100 m/s (see 2c)
0.25	62.5	90 m/s
0.1	48.4	84 m/s
0.05	44.10	82 m/s
0.01	40.804	80.4 m/s
0.005	40.401	80.2 m/s
0.001	40.08004	80.04 m/s
⋮	⋮	⋮
~ 0	~ 40	$v(1) = ?$ 80 m/s

$\frac{62.5 - 40}{0.25} = 4(22.5) = 90$

$\frac{48.4 - 40}{0.1} = 10(8.4) = 84$

etc.

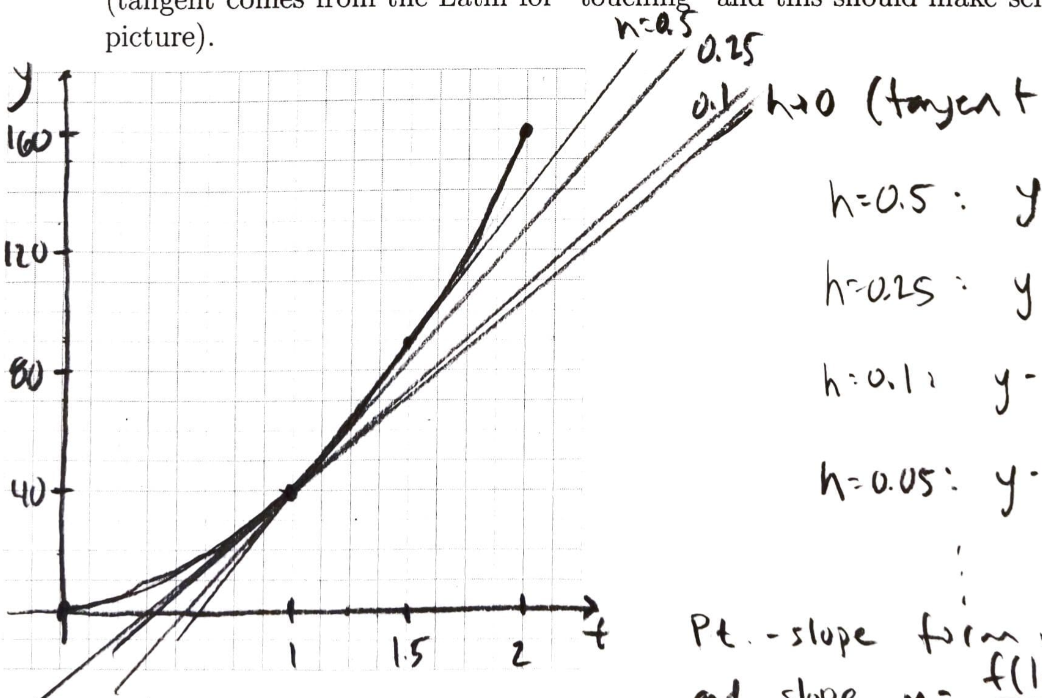
$(1+h, f(1+h))$

4. Sketch the following curves and lines below.

a) Draw the curve $y = f(t)$ over the interval $[0, 2]$ carefully (remember that this is just $y = 40t^2$).

b) Draw the lines through the points $(1, f(1))$ and $(1, f(1 + h))$ for $h = 0.5, 0.25, 0.1,$ ~~0.05~~. The slopes of these lines are the corresponding average velocities.

c) Plot the line through the point $(1, 40)$ with slope $v(1)$. This is a **tangent line** to the curve $y = f(t)$ (tangent comes from the Latin for "touching" and this should make sense after you've drawn the picture).



$h = 0.5$: $y - 40 = 100(t - 1)$

$h = 0.25$: $y - 40 = 90(t - 1)$

$h = 0.1$: $y - 40 = 84(t - 1)$

$h = 0.05$: $y - 40 = 82(t - 1)$

Pt.-slope form with point $(1, 40)$ and slope $m = \frac{f(1+h) - 40}{h}$ = avg velocity from the table in problem #3.