Definition. A function f is **continuous** at c if $\boxed{\lim_{x\to c} f(x) = f(c)}$. This means that f is only continuous at c if all 3 of the following are true:

(1) $f(c)$ is defined,
(2) $\lim_{x\to c} f(x)$ exists, and
(3) $\lim_{x\to c} f(x) = f(c)$.

1. Explain (with reference to the list above) why each of the following functions is **not** continuous at $c = 2$.

a) $f(x) = \dfrac{1}{2-3x+x^2}$

$f(2)$ is not defined (item 1)

Also $\lim_{x\to 2} f(x)$ DNE (item 2)

b) $g(x) = \dfrac{2-x}{2-3x+x^2}$

$f(2)$ is not defined (item 1)

Note $\dfrac{2-x}{2-3x+x^2} = \dfrac{2-x}{(2-x)(1-x)} = \dfrac{1}{1-x}$ if $x \neq 2$.

This means $\lim_{x\to 2} g(x) = \lim_{x\to 2} \dfrac{1}{1-x} = \dfrac{1}{-1} = -1$

c) $h(x) = \begin{cases} \frac{2-x}{2-3x+x^2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

$\lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{2-x}{2-3x+x^2} = \lim_{x\to 2} \dfrac{1}{1-x} = -1$

$h(2) = 1 \neq -1$.

item 3 fails.

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2. What value c would make the function continuous at 0?

a) $f(x) = \begin{cases} c - x^2 & \text{if } x \geq 0 \\ \cos x & \text{if } x < 0 \end{cases}$

Require $\lim_{x \to 0^+} c - x^2 = \lim_{x \to 0^-} \cos x = 1$

$\lim_{x \to 0^+} c - x^2 = c$. Thus $c = 1$ works.

b) $g(x) = \begin{cases} \frac{\sqrt{4+x}-2}{x} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$

Require $\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = c$

$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \to 0} \frac{x}{x(\sqrt{4+x} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{4}$

Thus $c = \frac{1}{4}$ works.

3. Find the points at which the function is **not** continuous.

a) $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^3 & \text{if } -1 \leq x < 1 \\ \frac{1}{x} & \text{if } 1 \leq x \end{cases}$

Continuous everywhere

$\left.\begin{array}{l} \lim_{x \to -1^-} x = -1 \\ \lim_{x \to -1} x^3 = -1 \end{array}\right\}$ $\lim_{x \to -1} f(x) = -1$ and $f(-1) = (-1)^3 = -1$ ✓

$\lim_{x \to 1^-} x^3 = 1$ $\lim_{x \to 1^+} \frac{1}{x} = 1$ and $f(1) = \frac{1}{1} = 1$. ✓

b) $g(x) = \dfrac{1}{1 - \cos x}$

Discontinuous when $\cos x = 1$: $x = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

c) How does your answer to part b change if we restrict the domain of g to the interval $(0, \pi)$?

Then the function is continuous everywhere (on the domain)