SEQUENCES AND SERIES

Definitions:

- A **sequence** $\{a_n\}$ is a list of numbers: $a_1, a_2, a_3, a_4, \dots$
- A **series** is a sum of numbers: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$.
- There are two sequences associated with the series $\sum_{n=1}^{\infty} a_n$:
  The sequence of terms: $a_1, a_2, a_3, \dots$.
  The sequence of **partial sums**: $s_1, s_2, s_3 \dots$ where $s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$.
- The **sum of a series** is the limit of the sequence of partial sums: $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k$.
- The series $\sum a_n$ is **absolutely convergent** if both $\sum a_n$ and $\sum |a_n|$ are convergent.
- The series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Tests:

- **Test for divergence**: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.
- **Integral test**: if f is a positive, decreasing, continuous function on the interval $[N, \infty)$ and $a_n = f(n)$ for $n \geq N$, then the series $\sum a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.
- **Direct comparison**: suppose that $0 \leq a_n \leq b_n$ for all $n \geq N$.
  If $\sum b_n$ converges, then $\sum a_n$ converges.
  If $\sum a_n$ diverges, then $\sum b_n$ diverges.
- **Limit comparison test**: suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$.
  If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
  If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
- **Ratio test**: let $L = \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$
  If $L < 1$, then the series $\sum a_n$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_n$ is divergent.
  If $L = 1$, then no conclusion can be drawn.
- **Root test**: let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$
  If $L < 1$, then the series $\sum a_n$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_n$ is divergent.
  If $L = 1$, then no conclusion can be drawn.
- **Alternating series test**: the alternating series $\sum (-1)^n b_n$ converges if:
  i) $0 \leq b_{n+1} \leq b_n$ for all $n \geq N$ and
  ii) $\lim_{n\to\infty} b_n = 0$,

Special series:

- **Geometric series:** $\displaystyle\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ if $|r| < 1$ (and diverges if $|r| \geq 1$).
- The p**-series** $\displaystyle\sum \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$ is convergent if and only if $p > 1$.
- The **harmonic series** $\displaystyle\sum \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent (p-series with $p = 1$).

Date: October 27, 2020.