MATH 258 EXAM 2 OCTOBER 10, 2012

NAME:

INSTRUCTIONS: Answer the questions in the space provided. Show all your work: even a correct answer may receive little or no credit if a method of solution is not shown. Solutions for indefinite integrals should be in terms of the original variable and without the composition of trig and inverse trig functions ($\sin(\cos^{-1} x)$ and other such expressions will not earn full credit).

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \qquad \cos^2 u = \frac{1}{2}(1 + \cos 2u) \qquad \sin 2u = 2(\sin u \cos u)$$

$$\int \tan u \, du = \ln|\sec u| + C \qquad \int \sec u \, du = \ln|\sec u + \tan u| + C \qquad \int \frac{du}{u^x + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

1. Evaluate the integral $\int_0^1 x \cos \pi x \, dx$.

$u = x \qquad dv = \cos \pi x \, dx$

$du = dx \qquad v = \frac{1}{\pi} \sin \pi x$

$$\int_0^1 x \cos \pi x \, dx = \frac{x}{\pi} \sin \pi x \Big|_0^1 - \int_0^1 \frac{1}{\pi} \sin \pi x \, dx$$

$$= [0 - 0] + \left[\frac{1}{\pi^2} \cos \pi x\right]_0^1$$

$$= \frac{1}{\pi^2}(-1) - \frac{1}{\pi^2}(1)$$

$$= -\frac{2}{\pi^2}$$

2. Evaluate the integral $\int \frac{\ln x}{x^2} \, dx$.

$u = \ln x \qquad dv = \frac{1}{x^2} dx$

$du = \frac{1}{x} dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

3. Evaluate the integral $\int \sin^3 t\, dt$.

$$\int \sin^3 t\, dt = \int (\sin^2 t) \sin t\, dt = \int (1-\cos^2 t) \sin t\, dt \qquad \begin{aligned} u &= \cos t \\ -du &= \sin t\, dt \end{aligned}$$

$$= \int (1-u^2)(-1)\, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 t}{3} - \cos t + C$$

4. Evaluate the integral $\int_0^2 \sqrt{16-x^2}\, dx$.

Let $x = 4\sin\theta$. Then $dx = 4\cos\theta\, d\theta$ and $x=0$ corresponds to $\theta=0$ while $x=2$ corresponds to $\theta = \sin^{-1}(\frac{2}{4}) = \frac{\pi}{6}$.

$$\int_0^2 \sqrt{16-x^2}\, dx = \int_0^{\frac{\pi}{6}} \sqrt{16-(4\sin\theta)^2}\, 4\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} 16\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 8(1+\cos 2\theta)\, d\theta$$

$$= 8\theta + 4\sin 2\theta \Big|_0^{\pi/6}$$

$$= 8\left(\frac{\pi}{6}\right) + 4\sin\left(\frac{2\pi}{6}\right) - (0+0)$$

$$= \frac{4\pi}{3} + 2\sqrt{3}$$

5. Evaluate the integral $\int \frac{\sqrt{x^2-1}}{x}\,dx$.

Let $x = \sec\theta$. Then $dx = \sec\theta\tan\theta\,d\theta$.

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$$

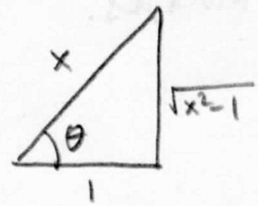
$$= \int \frac{\tan^2\theta\sec\theta}{\sec\theta}\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{x^2-1} - \sec^{-1}x + C$$



$\sec\theta = x \Leftrightarrow \cos\theta = \frac{1}{x}$

6. Evaluate the integral $\int \frac{4x+1}{x(x+1)^2}\,dx$.

$$\frac{4x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$

$$4x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A$$

$0 = A+B \Rightarrow \underline{B=-1}$

$4 = 2A+B+C \Rightarrow 4 = 2-1+C \Rightarrow \underline{C=3}$

$\underline{1 = A}$

$$\int \frac{4x+1}{x(x+1)^2}\,dx = \int \frac{1}{x} - \frac{1}{x+1} + \frac{3}{(x+1)^2}\,dx = \ln|x| - \ln|x+1| - \frac{3}{x+1} + C$$

7. Determine if the improper integral converges or diverges. If it converges evaluate it. $\int_0^1 \frac{e^{\frac{1}{x}}}{x^2}\,dx$.

$$\int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} -e^{1/x}\Big|_t^1$$

$$= \lim_{t\to 0^+} e^{1/t} - e$$

$$= \infty$$

The integral diverges.

8. Use the comparison theorem to determine if the improper integral converges or diverges. Be very clear about the comparison you are using. Do not evaluate the integral. $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$.

$$x^3+1 \geq x^3$$

$$\sqrt{x^3+1} \geq \sqrt{x^3}$$

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

$\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges (because $\frac{3}{2} > 1$).

Therefore by the comparison test $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges.