

NAME: Solutions

INSTRUCTIONS: Answer the questions in the space provided. Show all your work: even a correct answer may receive little or no credit if a method of solution is not shown. Always specify which tests for convergence or divergence you are using and, where appropriate, which series are being used for comparisons.

Some important series and their radii of convergence:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$	$R = 1$

1. Determine if the sequence $\left\{ \cos \left(\frac{\pi}{n} \right) \right\}_{n=1}^{\infty}$ converges or diverges. If it converges, find its limit.

$$\lim_{n \rightarrow \infty} \cos \left(\frac{\pi}{n} \right) = \cos 0 = 1$$

The sequence converges to 1.

2. Determine whether the series $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$ is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = 3 \sum_{n=0}^{\infty} \left(\frac{3}{4} \right)^n = 3 \left(\frac{1}{1 - \frac{3}{4}} \right) = 3(4) = 12$$

The series is convergent and its sum is 12.

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+2}$ is convergent or divergent.

$$\frac{n^2}{n^3+2} \leq \frac{n^2}{n^3} = \frac{1}{n} \quad \text{and} \quad \sum \frac{1}{n} \text{ diverges (harmonic series).}$$

Direct comparison doesn't work. Try limit comparison.

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{n^2}{n^3+2}} = \lim_{n \rightarrow \infty} \frac{n^3+2}{n^3} = 1.$$

Therefore $\sum_{n=1}^{\infty} \frac{n^2}{n^3+2}$ diverges.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ is convergent or divergent.

$$\frac{1}{n(n+3)} \leq \frac{1}{n^2} \quad \text{and} \quad \sum \frac{1}{n^2} \text{ converges}$$

(p-series with $p=2>1$).

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ converges by comparison.

5. Determine whether the series $\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n+1}}$ is absolutely convergent, conditionally convergent, or divergent.

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1.$$

Hence $\lim_{n \rightarrow \infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n+1}}$ does not exist.

The series $\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{\sqrt{n+1}}$ diverges by the test for divergence.

6. Find the radius of convergence and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$.

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{2^n x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^2}{n+1} \right| = 0 \text{ for all } x.$

Therefore the series converges for all x .

The radius of convergence is ∞ and the interval of convergence is $\mathbb{R} = (-\infty, \infty)$.

7. Find a power series representation of the function $f(x) = \frac{1}{8-x^3}$ and determine the interval of convergence.

$$\frac{1}{8-x^3} = \frac{1}{8} \left(\frac{1}{1-\frac{x^3}{8}} \right) = \frac{1}{8} \sum_{n=0}^{\infty} \left(\frac{x^3}{8} \right)^n = \sum_{n=0}^{\infty} \frac{x^{3n}}{8^{n+1}}$$

$$\hookrightarrow \text{when } \left| \frac{x^3}{8} \right| < 1 \Leftrightarrow |x^3| < 8 \Leftrightarrow |x| < 2.$$

Radius of convergence: 2.

$$x=2: \sum_{n=0}^{\infty} \frac{2^{3n}}{8^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{8} \text{ diverges.}$$

$$x=-2: \sum_{n=0}^{\infty} \frac{(-2)^{3n}}{8^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{3n}}{8} \text{ diverges.}$$

Interval of convergence: $(-2, 2)$.

8. Use the binomial series to evaluate $\int \sqrt{1+x^2} dx$ as a series. Either write your answer using a \sum or find the first 4 terms of the series. Do not simplify your answer.

$$\begin{aligned}\sqrt{1+x^2} &= (1+x^2)^{1/2} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)(x^2)^n \quad \text{when } x^2 < 1, \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) x^{2n} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2} x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} x^6 + \dots \\ &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \dots \\ \int \sqrt{1+x^2} dx &= C + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \frac{x^{2n+1}}{2n+1} \\ &= C + x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{112}x^7 + \dots\end{aligned}$$

9. Find the Taylor series for $f(x) = \ln x$ at $a = 1$. Either write your answer using a \sum or find the first 5 terms of the series (i.e. find c_0, c_1, c_2, c_3 , and c_4 where $\ln x = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3 + c_4(x-1)^4 + \dots$).

$$c_n = \frac{f^{(n)}(1)}{n!}$$

$$f(x) = \ln x$$

$$f(1) = 0$$

$$c_0 = 0$$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = 1$$

$$c_1 = 1$$

$$f''(x) = -\frac{1}{x^2}$$

$$f''(1) = -1$$

$$c_2 = -\frac{1}{2}$$

$$f^{(3)}(x) = \frac{2}{x^3}$$

$$f^{(3)}(1) = 2$$

$$c_3 = \frac{2}{3!} = \frac{1}{3}$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$f^{(4)}(1) = -6$$

$$c_4 = -\frac{6}{4!} = -\frac{1}{4}$$

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n} \quad f^{(n)}(1) = (-1)^{n-1} (n-1)! \quad c_n = \frac{(-1)^{n-1}}{n!}$$

$$\ln x = x-1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} (x-1)^n$$