

**Directly?** Possibly after some algebra. Examples:

- $\int \frac{\sqrt{x}}{4+\sqrt{x}} dx$  not direct.  $\int \frac{x}{4+x} dx = \int 1 - \frac{4}{4+x} dx = x - 4 \ln|4+x| + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$  or  $-\cos^{-1} x + C$
- $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + C$  or  $\coth^{-1} x + C$
- $\int (x-1)(x+2) dx = \int x^2 + x - 2 dx = \frac{1}{3} x^3 + \frac{1}{2} x^2 - 2x + C$

**Partial Fractions?** Just fancy algebra. Watch for rational functions  $\frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomials.

First division. Different forms for partial fractions decomposition based on the factorization of the denominator (cases I-III are the most important, see pp. 323-326). Examples:

- $\int \frac{1}{x^2 + 3x - 4} dx = \int \frac{1}{5} \left( \frac{1}{x-1} - \frac{1}{x+4} \right) dx = \frac{1}{5} \ln|x-1| - \ln|x+4| + C$
- $\int \frac{x^3 - 4x - 10}{x^2 - x - 6} dx = \int x+1 + \frac{3x-4}{x^2-x-6} dx = \int x+1 + \frac{1}{x-3} + \frac{2}{x+2} dx \dots$
- $\int \frac{x^2 + 2x - 1}{x^3 - x} dx = \int \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1} dx = \dots$
- $\int \frac{x^2}{(x-3)(x+2)^2} dx = \int \frac{1}{25} \left[ \frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2} \right] dx = \dots$

**Substitution?**  $\int f(g(x))g'(x) dx = \int f(u) du$  where  $u = g(x)$ . Watch for compositions of functions. Examples:

- $\int \frac{1}{x \ln x} dx = \ln|\ln x| + C$
- $\int \frac{e^{\frac{1}{x}}}{x^2} dx = -e^{\frac{1}{x}} + C$
- $\int e^x \sin(e^x) dx = -\cos(e^x) + C$
- $\int (1+3x)^7 dx = \frac{1}{24} (1+3x)^8 + C$

**Integration by Parts?**  $\int u dv = uv - \int v du$ . Watch for products of functions. Examples:

- $\int x e^{-x} dx$ 
 $u = x$ 
 $dv = e^{-x} dx$ 
 $du = dx$ 
 $v = -e^{-x}$ 
 $-x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C$
- $\int x^2 \cos x dx$ 
 $u = x^2$ 
 $dv = \cos x dx$ 
 $du = 2x dx$ 
 $v = \sin x$ 
 $x^2 \sin x - \int 2x \sin x dx$ 
 $u = x$ 
 $dv = \sin x dx$ 
 $du = dx$ 
 $v = -\cos x$
- $\int \sqrt{x} \ln x dx$ 
 $u = \ln x$ 
 $dv = \sqrt{x} dx$ 
 $du = \frac{1}{x} dx$ 
 $v = \frac{2}{3} x^{3/2}$ 
 $\frac{2}{3} x^{3/2} \ln x - \int \frac{2}{3} x^{1/2} dx = \dots$
- $\int (x^2 + 1) \cos x dx = \int x^2 \cos x + \cos x dx = \dots$  see #2 above
- $\int \ln x dx$ 
 $u = \ln x$ 
 $dv = dx$ 
 $du = \frac{1}{x} dx$ 
 $v = x$ 
 $x \ln x - \int dx = x \ln x - x + C$

**Trig integral?** An integral involving *only* trig functions should be approached using the techniques outlined in the margins of pp. 314 and 315. Examples:

- $\int \sin^6 x \cos^3 x dx = \int \sin^6 (1 - \sin^2 x) \cos x dx$ 
 $u = \sin x$ 
 $du = \cos x dx$ 
 $\int u(1-u^2) du = \dots$
- $\int \sin^2(2x) dx = \int \frac{1 - \cos(4x)}{2} dx = \frac{1}{2} \left[ x - \frac{1}{4} \sin(4x) \right] + C$
- $\int \tan^4 x dx = \int (\sec^2 x - 1) \tan^2 x dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$
- $\int \sec x \tan^3 x dx$   
 $= \int (\sec^2 x - 1) \sec x \tan x dx$   
 $u = \sec x$ 
 $du = \sec x \tan x dx$

 $= \frac{1}{3} \tan^3 x - \int (\sec^2 x - 1) dx$   
 $= \frac{1}{3} \tan^3 x - \tan x + x + C$

**Trig substitution?** Three expressions to watch for:

- $\sqrt{a^2 - x^2}$  substitute  $x = a \sin \theta$ , identity  $1 - \sin^2 \theta = \cos^2 \theta$ ;
- $\sqrt{a^2 + x^2}$  substitute  $x = a \tan \theta$ , identity  $1 + \tan^2 \theta = \sec^2 \theta$ ;
- $\sqrt{x^2 - a^2}$  substitute  $x = a \sec \theta$ , identity  $\sec^2 \theta - 1 = \tan^2 \theta$ ;

You should end up with a trig integral (see above). Examples:

- $\int \frac{x^3}{\sqrt{16 - x^2}} dx$
- $\int x^3 \sqrt{x^2 + 4} dx$
- $\int \sqrt{1 - 9x^2} dx$
- $\int \frac{dx}{x^2 \sqrt{x^2 - 9}}$

# Trig Substitution Solutions

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1.  $\int \frac{x^3}{\sqrt{16-x^2}} dx$        $x = 4 \sin \theta$   
 $dx = 4 \cos \theta d\theta$

$$= \int \frac{64 \sin^3 \theta}{\sqrt{16-16 \sin^2 \theta}} 4 \cos \theta d\theta = \int 64 \sin^3 \theta d\theta$$

$$= \int 64(1-\cos^2 \theta) \sin \theta d\theta$$

$u = \cos \theta$   
 $du = -\sin \theta d\theta$

$$= \int -64(1-u^2) du$$

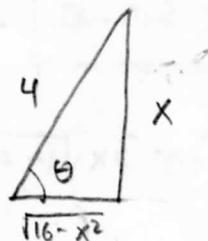
$$= -64(u - \frac{1}{3}u^3) + C$$

$$= 64(\frac{1}{3} \cos^3 \theta - \cos \theta) + C$$

$$= 64 \left[ \frac{1}{3} \left( \frac{\sqrt{16-x^2}}{4} \right)^3 - \frac{\sqrt{16-x^2}}{4} \right] + C$$

$$= \frac{1}{3}(16-x^2)\sqrt{16-x^2} - 16\sqrt{16-x^2} + C$$

$$= \boxed{-\left(\frac{32+x^2}{3}\right)\sqrt{16-x^2} + C}$$



$$\sin \theta = \frac{x}{4}$$

$$\cos \theta = \frac{\sqrt{16-x^2}}{4}$$

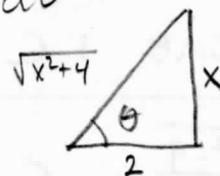
2.  $\int x^3 \sqrt{x^2+4} dx$        $x = 2 \tan \theta$        $dx = 2 \sec^2 \theta d\theta$

$$= \int 8 \tan^3 \theta \sqrt{4 \tan^2 \theta + 4} 2 \sec^2 \theta d\theta$$

$$= \int 32 \tan^3 \theta \sec^3 \theta d\theta = \int 32 (\sec^2 \theta - 1) \sec^2 \theta (\sec \theta \tan \theta) d\theta$$

u.

$$u = \sec \theta \quad du = \sec \theta \tan \theta d\theta \dots$$



$$\tan \theta = \frac{x}{2}$$
$$\sec \theta = \frac{\sqrt{x^2+4}}{2}$$

$$3. \int \sqrt{1-9x^2} dx \quad \text{want } 9x^2 = \sin^2 \theta \quad \text{so } x = \frac{1}{3} \sin \theta \quad dx = \frac{1}{3} \cos \theta d\theta$$

$$= \int \sqrt{1-\sin^2 \theta} \left(\frac{1}{3}\right) \cos \theta d\theta$$

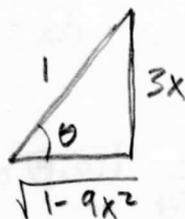
$$= \int \frac{1}{3} \cos^2 \theta d\theta$$

$$= \int \frac{1}{3} \left( \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{6} \left[ \theta + \frac{1}{2} \sin 2\theta \right] + C$$

use  $\sin 2\theta = 2 \sin \theta \cos \theta$

$$= \boxed{\frac{1}{6} \left[ \sin^{-1}(3x) + 3x\sqrt{1-9x^2} \right] + C}$$



$$\sin \theta = 3x$$

$$\cos \theta = \sqrt{1-9x^2}$$

$$\theta = \sin^{-1}(3x)$$

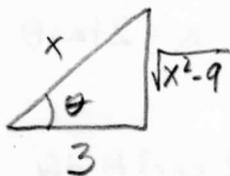
$$4. \int \frac{1}{x^2 \sqrt{x^2-9}} dx \quad x = 3 \sec \theta \quad dx = 3 \sec \theta \tan \theta d\theta$$

$$= \int \frac{1}{9 \sec^2 \theta \sqrt{9 \sec^2 \theta - 9}} 3 \sec \theta \tan \theta d\theta$$

$$= \int \frac{3 \sec \theta \tan \theta}{9 \sec^2 \theta \cdot 3 \tan \theta} d\theta = \int \frac{1}{9} \cos \theta d\theta$$

$$= \frac{1}{9} \sin \theta + C$$

$$= \boxed{\frac{\sqrt{x^2-9}}{9x} + C}$$



$$\sec \theta = \frac{x}{3}$$

$$\sin \theta = \frac{\sqrt{x^2-9}}{x}$$