A **sequence** $\{a_n\}$ is a list of numbers: $a_1, a_2, a_3, \ldots, a_n, \ldots$. A **series** $\sum a_n$ is a sum of numbers: $a_1 + a_2 + a_3 + \cdots + a_n + \ldots$. There are two sequences associated with the series $\sum a_n$:

1. the sequence of terms: $a_1, a_2, a_3, \ldots$ and
2. the sequence of partial sums: $s_1, s_2, s_3 \ldots$ where $s_n = a_1 + a_2 + a_3 + \cdots + a_n$.

The sum of a series is the limit of the sequence of partial sums: $\sum a_n = \lim_{n\to\infty} s_n$.

Tests:

- Test for divergence: if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum a_n$ diverges. Without symbols: if the limit of the sequence of terms is not zero, then the series diverges.
- Integral test: suppose f is a positive, decreasing function on the interval $[b, \infty)$. Then the series $\sum a_n$ is convergent if and only if the improper integral $\int_b^\infty f(x)\,dx$ is convergent.
- Comparison test: suppose that $0 \leq a_n \leq b_n$ for all n.

  If $\sum b_n$ converges, then $\sum a_n$ converges.

  If $\sum a_n$ diverges, then $\sum b_n$ diverges.
- Limit comparison test: suppose that $a_n > 0$ and $b_n > 0$ for all n and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$.

  If $0 < L < \infty$, then $\sum s_n$ converges if and only if $\sum b_n$ converges.

  If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.

  If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- Alternating series test: if the alternating series $\sum (-1)^n b_n$ satisfies
  1. $b_{n+1} \leq b_n$ for all n and
  2. $\lim_{n\to\infty} b_n = 0$,

  then the series converges.
- Ratio test: let $L = \lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$.

  If $L < 1$, then the series $\sum a_n$ is absolutely convergent.

  If $L > 1$, then the series $\sum a_n$ is divergent.

  If $L = 1$, then no conclusion can be drawn.
- Root test: let $L = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

  If $L < 1$, then the series $\sum a_n$ is absolutely convergent.

  If $L > 1$, then the series $\sum a_n$ is divergent.

  If $L = 1$, then no conclusion can be drawn.

Miscellaneous:

- The **geometric series** $\sum_{n=0}^\infty x^n$ converges if and only if $|x| < 1$ in which case $\displaystyle\sum_{n=0}^\infty x^n = \frac{1}{1-x}$.
- A p**-series** $\sum \frac{1}{n^p}$ is convergent if and only if $p > 1$.
- The Taylor series of a function f at a is $\displaystyle\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$.
- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum c_n (x-a)^n\right] = \sum \left[\frac{\mathrm{d}}{\mathrm{d}x} c_n (x-a)^n\right]$.
- $\displaystyle\int \left[\sum c_n (x-a)^n\right] dx = C + \sum \left[\int c_n (x-a)^n \, dx\right]$.