NAMES: solutions
MATH 258 FROM FUNCTIONS TO POWER SERIES NOVEMBER 24, 2015

$$\boxed{\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \text{ if } |u| < 1}$$

1. Use the formula above to express $\dfrac{3}{1-2x}$ as a power series. What is the radius of convergence of the power series?

$$\frac{3}{1-2x} = 3\sum_{n=0}^{\infty}(2x)^n = 3\sum_{n=0}^{\infty}2^n x^n \quad \text{if } |2x|<1$$

$|2x|<1$ if and only if $|x|<\frac{1}{2}$.

The radius of convergence is $\frac{1}{2}$.

2. The goal of this problem is to find two different power series representations of $\dfrac{1}{2-x}$.

a) Use the expression $\dfrac{1}{2-x} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{1-\frac{x}{2}}\right)$ and the formula at the top of the page to obtain a power series representation for $\dfrac{1}{2-x}$.

$$\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}} \quad \text{if } \left|\frac{x}{2}\right|<1$$

b) Use the expression $\dfrac{1}{2-x} = \dfrac{1}{1-(x-1)}$ to obtain a different power series representation for $\dfrac{1}{2-x}$.

$$\frac{1}{2-x} = \sum_{n=0}^{\infty}(x-1)^n \quad \text{if } |x-1|<1$$

c) What are the radii of convergence of the two power series?

$\left|\frac{x}{2}\right|<1$ if and only if $|x|<2$. The R.O.C. for a is 2.

$|x-1|<1$ means the R.O.C. for b is 1.

The interval of convergence for b is $(0,2)$.

Theorem 2 in the book says that derivatives and integrals work for power series. Consequently, if $|x| < 1$, then

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \ldots\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

and

$$-\ln|1-x| + C = \int \frac{1}{1-x}dx = \int \left(\sum_{n=0}^{\infty} x^n\right) dx = \int \left(1 + x + x^2 + x^3 + \ldots\right) dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

for some value of C (and it turns out that $C = 0$).

3. We know that if $|x| < 1$, then $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$.

a) Integrate both sides of this equation to find a power series representation of $\tan^{-1} x$ (you'll need to determine the value of the constant that appears when you integrate: evaluate at $x = 0$ to do this).

$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Let $x=0$: $0 = C + 0 + 0 + \cdots$

Thus $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

b) We know that your series converges when $|x| < 1$. Show that it also converges for $x = 1$.

If $x=1$ then we have $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$

Apply the alternating series test: $b_n = \frac{1}{2n+1}$

i) $b_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} \leq \frac{1}{2n+1} = b_n$ ✓

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓

The series converges.

c) Use your power series the fact that $\pi = 4\tan^{-1}(1)$ to find a series whose sum is π.

$\pi = 4\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$