NAMES: solutions

MATH 258 USING POWER SERIES DECEMBER 2, 2015

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \text{ for all } u$$

1. Recall that $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$.

a) Use the power series expansions of e^x and e^{-x} to find power series expansions of $\sinh x$ and $\cosh x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!},$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

b) Use your power series expansion of $\sinh x$ to help evaluate the limit $\lim_{x\to 0} \frac{\sinh x}{x}$ (without using l'Hospital's rule).

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{1}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \lim_{x\to 0} 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$$

$$= 1$$

c) Use your power series expansion of $\cosh x$ to help evaluate the limit $\lim_{x\to 0} \frac{1-\cosh x}{x}$ (without using l'Hospital's rule).

$$\lim_{x\to 0} \frac{1-\cosh x}{x} = \lim_{x\to 0} \frac{1}{x}\left[1 - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$$= \lim_{x\to 0} -\frac{x}{2!} - \frac{x^3}{4!} - \cdots$$

$$= 0$$

Note: $\lim_{x\to 0} \frac{1-\cosh x}{x^2} = \lim_{x\to 0} -\frac{1}{2} - \frac{x^2}{4!} - \frac{x^4}{6!} \cdots = -\frac{1}{2}.$

$$(1+u)^k = 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \frac{k(k-1)(k-2)(k-3)}{4!}u^4 + \ldots \text{ if } |u| < 1$$

2. Recall that $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$.

a) Use the binomial series to find a power series expansion for $\frac{1}{\sqrt{1+x^2}}$ (there is no need to find a nice pattern–just write out at least the first 4 terms of the series).

$$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^4 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^6 + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots \quad \text{if } |x| < 1$$

b) Anti-differentiate your series to find a power series for $\sinh^{-1} x$ (again, there is no need to find a nice pattern).

$$\sinh^{-1} x = C + x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \frac{35}{1152}x^8 + \cdots$$

Let $x = 0$; $\sinh^{-1} 0 = C + 0 + 0 + \cdots$ so $C = 0$

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \frac{35}{1152}x^8 + \cdots \quad \text{if } |x| < 1$$

c) Use your power series to find and approximation to $\sinh^{-1}\left(\frac{1}{2}\right)$.

$$\sinh^{-1}\left(\tfrac{1}{2}\right) \approx \frac{1}{2} - \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5 - \frac{5}{112}\left(\frac{1}{2}\right)^7 + \frac{35}{1152}\left(\frac{1}{2}\right)^8 \approx 0.48128$$

Calculator: $\sinh^{-1}\left(\tfrac{1}{2}\right) \approx 0.4812118250\!6$