IMPROPER INTEGRALS

Definition. Let $f(x)$ be any function.

i) For any number a,

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

ii) For any number b,

$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

Also,

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx$$

for any number c. In this case, the whole integral diverges if either piece on the right hand side diverges.

Definition. If $f(x)$ has a vertical asymptote at $x = b$, then …

i) …for any number $a < b$,

$$\int_a^b f(x)\ dx = \lim_{c\to b^-} \int_a^c f(x)\ dx$$

ii) …for any number $a > b$,

$$\int_b^a f(x)\ dx = \lim_{c\to b^+} \int_c^a f(x)\ dx$$

Also, if $a < b < c$, then

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx$$

In this case, the whole integral diverges if either piece on the right hand side diverges.

Theorem (Comparisons)**.** *If $0 \le f(x) \le g(x)$, then …*

i) …if $\int_a^b f(x)\ dx$ diverges to infinity, then $\int_a^b g(x)\ dx$ also diverges to infinity;
ii) …if $\int_a^b g(x)\ dx$ converges, then $\int_a^b f(x)\ dx$ also converges.

Theorem (Limit Comparisons)**.** *If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ and $0 < L < \infty$, then either $\displaystyle\int_a^\infty f(x)\ dx$ and $\displaystyle\int_a^\infty g(x)\ dx$ …*

i) both converge, or …
ii) both diverge.

Theorem (Basic integrals to use in comparisons)**.**

$$\int_1^\infty \frac{1}{x^p}\ dx \ldots$$

i) …converges if $p > 1$;
ii) …diverges if $p \le 1$.

$$\int_0^1 \frac{1}{x^p}\ dx \ \ldots$$

i) …converges if $p < 1$;
ii) …diverges if $p \ge 1$.

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