

To find  $A$  and  $B$ , we first clear Equation (1) of fractions and regroup in powers of  $x$ , obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in  $x$  if and only if the coefficients of like powers of  $x$  on the two sides are equal:

$$A + B = 5, \quad -3A + B = -3.$$

Solving these equations simultaneously gives  $A = 2$  and  $B = 3$ .

### General Description of the Method

Success in writing a rational function  $f(x)/g(x)$  as a sum of partial fractions depends on two things:

- *The degree of  $f(x)$  must be less than the degree of  $g(x)$ .* That is, the fraction must be proper. If it isn't, divide  $f(x)$  by  $g(x)$  and work with the remainder term. Example 3 of this section illustrates such a case.
- We must know the factors of  $g(x)$ . In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction  $f(x)/g(x)$  when the factors of  $g$  are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

#### Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let  $x - r$  be a linear factor of  $g(x)$ . Suppose that  $(x - r)^m$  is the highest power of  $x - r$  that divides  $g(x)$ . Then, to this factor, assign the sum of the  $m$  partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of  $g(x)$ .

2. Let  $x^2 + px + q$  be an irreducible quadratic factor of  $g(x)$  so that  $x^2 + px + q$  has no real roots. Suppose that  $(x^2 + px + q)^n$  is the highest power of this factor that divides  $g(x)$ . Then, to this factor, assign the sum of the  $n$  partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of  $g(x)$ .

3. Set the original fraction  $f(x)/g(x)$  equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of  $x$ .
4. Equate the coefficients of corresponding powers of  $x$  and solve the resulting equations for the undetermined coefficients.

**EXAMPLE 1** Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx.$$