POWER SERIES

Basics:

- A **sequence** $\{a_n\}$ is a list of numbers: $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$.
- A **series** is a sum of numbers: $\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \ldots$.
- A **power series** centered at a is an infinite series of the form:

$$\sum_{k=1}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

  The coefficients c_k must be constant.

  The series converges for all x in an interval centered at a (and diverges for x outside that interval). This is the **interval of convergence**.

  The **radius of convergence** is:

  0 if the series conveges only for $x = a$

  A number $R > 0$ if the interval of convergence is $(a-R, a+R)$ (possibly including $a-R$ and/or $a+R$; convergence for $x = a-R$ and $x = a+R$ usually needs to be checked by hand)

  ∞ if the interval of convergence is $(-\infty, \infty)$

- There are two sequences associated with the series $\sum_{k=0}^{\infty} a_k$:

  The sequence of terms: $a_0, a_1, a_2, a_3, \ldots$.

  The sequence of **partial sums**: $s_1, s_2, s_3 \ldots$ where $s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + a_3 + \cdots + a_n$.

- The **sum of a series** is the limit of the sequence of partial sums: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$.
- The series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} |a_k|$ are convergent.
- The series $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent** if $\sum_{k=1}^{\infty} a_k$ is convergent, but $\sum_{k=1}^{\infty} |a_k|$ is divergent.

Power series:

- $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$ if $|x| < 1$
- $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ for all x
- $\cos x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^4}{4!} + \ldots$ for all x
- $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ for all x
- $\tan^{-1} x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7!} + \ldots$ if $|x| \le 1$
- $\ln x = \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$ if $|x-1| < 1$
- $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$.

Date: April 15, 2019.

Tests:

- **Test for divergence**: if $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ diverges.

- **Integral test**: if f is a positive, decreasing, continuous function on the interval $[b, \infty)$ and $a_k = f(k)$ for $k \geq b$, then the series $\sum a_k$ and the integral $\int_b^\infty f(x)\, dx$ are both convergent or both divergent.

- **Comparison test**: suppose that $0 \leq a_k \leq b_k$ for all $k \geq N$.
  If $\sum b_k$ converges, then $\sum a_k$ converges.
  If $\sum a_k$ diverges, then $\sum b_k$ diverges.

- **Limit comparison test**: suppose that $a_k > 0$ and $b_k > 0$ for all $k \geq N$ and $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = L$.
  If $0 < L < \infty$, then $\sum a_k$ and $\sum b_k$ are both convergent or both divergent.
  If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
  If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

- **Alternating series test**: the alternating series $\sum (-1)^k b_k$ converges if
  1) $0 \leq b_{k+1} \leq b_k$ for all k and
  2) $\lim_{k\to\infty} b_k = 0$,

- **Ratio test**: let $L = \displaystyle\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$.
  If $L < 1$, then the series $\sum a_k$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_k$ is divergent.
  If $L = 1$, then no conclusion can be drawn.

- **Root test**: let $L = \displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|}$.
  If $L < 1$, then the series $\sum a_k$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_k$ is divergent.
  If $L = 1$, then no conclusion can be drawn.

Special series:

- **Geometric series:**
  $$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \text{ if } |r| < 1$$
  $$\sum_{k=1}^{\infty} r^k = r + r^2 + r^3 + r^4 + \cdots = \frac{r}{1-r} \text{ if } |r| < 1$$

- The p-**series** $\displaystyle\sum \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ is convergent if and only if $p > 1$.

- The **harmonic series** $\displaystyle\sum \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent (p-series with $p = 1$).

Practice:

1. Find the radius of convergence and interval of convergence of the series:

   a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$

   b) $\displaystyle\sum_{k=1}^{\infty} kx^k$

   c) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k3^k}$

   d) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2x^k}{10^k}$

   e) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^k}$

   f) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k(x-3)^k}{k+4}$

2. Find a power series representation for the function and determine the radius of convergence:

   a) $f(x) = \ln(1+x)$

   b) $f(x) = \dfrac{1}{(1+x)^2}$

   c) $f(x) = \dfrac{x}{(x-1)^3}$

   d) $f(x) = \dfrac{3}{(2-x)^2}$

3. Evaluate as a power series:

   a) $\displaystyle\int \frac{1}{1+x^4}\, dx$

   b) $\displaystyle\int_0^{\frac{1}{2}} \tan^{-1}(x^2)\, dx$

   c) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x}{x}$

4. Find the Taylor series for f centered at the given value:

   a) $f(x) = \dfrac{1}{x},\ a = 1$

   b) $f(x) = \sin x,\ a = \dfrac{\pi}{4}$

   c) $f(x) = \sqrt{x},\ a = 4$

5. Find the sum of the series:

   a) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{k!}$

   b) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{5^k k!}$

   c) $\ln x + \dfrac{(\ln x)^2}{3} + \dfrac{(\ln x)^3}{9} + \dfrac{(\ln x)^4}{27} + \dfrac{(\ln x)^5}{81} + \dots$