SEQUENCES AND SERIES

Definitions:

- A **sequence** $\{a_n\}$ is a list of numbers: $a_1, a_2, a_3, \ldots, a_n, \ldots$.
- A **series** is a sum of numbers: $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$.
- There are two sequences associated with the series $\sum_{k=1}^{\infty} a_k$:
  The sequence of terms: $a_1, a_2, a_3, \ldots$.
  The sequence of **partial sums**: $s_1, s_2, s_3 \ldots$ where $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$.
- The **sum of a series** is the limit of the sequence of partial sums: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$.
- The series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} |a_k|$ are convergent.
- The series $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent** if $\sum_{k=1}^{\infty} a_k$ is convergent, but $\sum_{k=1}^{\infty} |a_k|$ is divergent.

Tests:

- **Test for divergence**: if $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ diverges.
- **Integral test**: if f is a positive, decreasing, continuous function on the interval $[b, \infty)$ and $a_k = f(k)$ for $k \geq b$, then the series $\sum a_k$ and the integral $\int_b^\infty f(x)\, dx$ are both convergent or both divergent.
- **Comparison test**: suppose that $0 \leq a_k \leq b_k$ for all $k \geq N$.
  If $\sum b_k$ converges, then $\sum a_k$ converges.
  If $\sum a_k$ diverges, then $\sum b_k$ diverges.
- **Limit comparison test**: suppose that $a_k > 0$ and $b_k > 0$ for all $k \geq N$ and $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L$.
  If $0 < L < \infty$, then $\sum a_k$ and $\sum b_k$ are both convergent or both divergent.
  If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
  If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.
- **Alternating series test**: the alternating series $\sum (-1)^k b_k$ converges if
  1) $0 \leq b_{k+1} \leq b_k$ for all k and
  2) $\lim_{k\to\infty} b_k = 0$,
- **Ratio test**: let $L = \lim\limits_{k\to\infty} \left| \dfrac{a_{k+1}}{a_k} \right|$.
  If $L < 1$, then the series $\sum a_k$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_k$ is divergent.
  If $L = 1$, then no conclusion can be drawn.
- **Root test**: let $L = \lim\limits_{k\to\infty} \sqrt[k]{|a_k|}$.
  If $L < 1$, then the series $\sum a_k$ is absolutely convergent.
  If $L > 1$, then the series $\sum a_k$ is divergent.
  If $L = 1$, then no conclusion can be drawn.

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Special series:

- **Geometric series:**
$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \text{ if } |r| < 1$$
$$\sum_{k=1}^{\infty} r^k = r + r^2 + r^3 + r^4 + \cdots = \frac{r}{1-r} \text{ if } |r| < 1$$
- The p**-series** $\sum \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ is convergent if and only if $p > 1$.
- The **harmonic series** $\sum \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent (p-series with $p = 1$).