SEQUENCES

Definition. The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there is a number N such that $|a_n - L| < \epsilon$ whenever $n > N$. If for some positive number ϵ no such number N exists, then the sequence does not converge to L. If the sequence does not converge to any number L, then the sequence **diverges**. If for every number M there is a number N such that $a_n > M$ whenever $n > N$, then the sequence **diverges to infinity**. If for every number M there is a number N such that $a_n < M$ whenever $n > N$, then the sequence **diverges to negative infinity**.

Notation.

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L$$
$$\lim_{n\to\infty} a_n = \infty \text{ or } a_n \to \infty$$
$$\lim_{n\to\infty} a_n = -\infty \text{ or } a_n \to -\infty$$

Theorem. *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.*

(1) $\lim_{n\to\infty}(a_n + b_n) = A + B$
(2) $\lim_{n\to\infty}(a_n - b_n) = A - B$
(3) $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ *(any number k)*
(4) $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
(5) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{A}{B}$ *(if $B \neq 0$)*

Theorem (Squeeze theorem)**.** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences. If $a_n \leq b_n \leq c_n$ for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$*

Theorem. *Let $\{a_n\}$ be a sequence and let f be a function.*

(1) If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.
(2) If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Theorem (Some important limits)**.**

(1) $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

(2) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$

(3) $\displaystyle\lim_{n\to\infty} x^{1/n} = 1 \ (x > 0)$

(4) $\displaystyle\lim_{n\to\infty} x^n = 0 \ (|x| < 1)$

(5) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ *(any x)*

(6) $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ *(any x)*

In formulas 3–6, x remains fixed as $n \to \infty$.

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1. Determine if each sequence converges or diverges. Find the limit of any convergent sequences.

a) $a_n = \dfrac{n + (-1)^n}{n}$

b) $a_n = \dfrac{1 - n^3}{100 - 4n^2}$

c) $a_n = (-1)^n \left(1 - \dfrac{1}{n}\right)$

d) $a_n = \dfrac{\sin^2 n}{n^2}$

e) $a_n = \dfrac{\ln(n)}{\ln(2n)}$

f) $a_n = \dfrac{(2n-1)!}{(2n+1)!}$