258 F20 WS-06 Solutions

1. a) $L = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$

b) $\int \sqrt{1+4x^2}\,dx = \int \sqrt{1+\tan^2\theta}\,\left(\frac{1}{2}\sec^2\theta\right)d\theta$

$2x = \tan\theta$

$dx = \frac{1}{2}\sec^2\theta\,d\theta$

$= \int \frac{1}{2}\sec^3\theta\,d\theta$

c) $\int \frac{1}{2}\sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta - \int \frac{1}{2}\sec\theta\tan^2\theta\,d\theta$

IBP $u = \frac{1}{2}\sec\theta$; $du = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$dv = \sec^2\theta\,d\theta$ $v = \tan\theta$

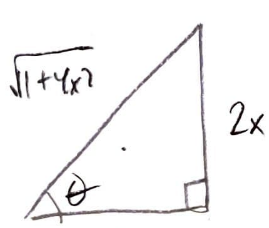
d) $= \frac{1}{2}\sec\theta\tan\theta - \int \frac{1}{2}\sec\theta(\sec^2\theta - 1)\,d\theta$

$= \frac{1}{2}\sec\theta\tan\theta - \int \frac{1}{2}\sec^3\theta\,d\theta + \int \frac{1}{2}\sec\theta\,d\theta$

Therefore

$\int \frac{1}{2}\sec^3\theta\,d\theta = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$

e) $= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x^2} + 2x\right|\right) + C$



$\tan\theta = 2x$

f) $\int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x} + 2x\right|\right)\Big|_0^1$

$= \frac{1}{4}\left[2\sqrt{5} + \ln\left|\sqrt{5}+2\right| - \left(0 + \ln\left|1+0\right|\right)\right]$

$= \frac{1}{4}\left(2\sqrt{5} + \ln\left(\sqrt{5}+2\right)\right)$

Verifying that $\frac{d}{dx}\left[\frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x} + 2x\right|\right)\right] = \sqrt{1+4x^2}$

is very difficult.

An arc length function is $s(x) = \int_0^x \sqrt{1+4t^2}\,dt$

$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x} + 2x\right|\right)$