Solutions to WS-07 Improper Integrals

1. a) $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4^-} \int_0^b \frac{1}{\sqrt{4-x}}\,dx$

$$= \lim_{b\to 4^-} -2\sqrt{4-x}\,\Big|_0^b$$

$$= \lim_{b\to 4^-} 4 - 2\underbrace{\sqrt{4-b}}_{\to 0^+}$$

$$= 4 \quad \text{Convergent}$$

b) $\int_1^2 \frac{1}{x\ln x}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{1}{x\ln x}\,dx$ $\qquad u = \ln x$, $du = \frac{1}{x}dx$

$$= \lim_{a\to 1^+} \int_{\ln a}^{\ln 2} \frac{1}{u}\,du$$

$$= \lim_{a\to 1^+} \ln|u|\,\Big|_{\ln a}^{\ln 2}$$

$$= \lim_{a\to 1^+} \ln|\ln 2| - \underbrace{\ln|\underbrace{\ln a}_{\to 0^+}|}_{\to -\infty}$$

$$= \infty \quad \text{Divergent}$$

c) $\displaystyle\int_2^\infty \frac{4}{\sqrt{x^3-1}}\,dx$

Limit comparison with $\frac{1}{x^{3/2}}$:

$$\lim_{x\to\infty} \frac{1/x^{3/2}}{4/\sqrt{x^3-1}} = \lim_{x\to\infty} \frac{\sqrt{x^3-1}}{4x^{3/2}} = \lim_{x\to\infty} \frac{1}{4}\sqrt{1-\frac{1}{x^3}} = \frac{1}{4}$$

$\frac{1}{x^3} \to 0$

Therefore $\displaystyle\int_2^\infty \frac{4}{\sqrt{x^3-1}}\,dx$ and $\displaystyle\int_2^\infty \frac{1}{x^{3/2}}\,dx$ do the same thing, namely converge.

d) $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx$$

By symmetry, both integrals on the right hand side do the same thing. Also, only the tails matter, since $y=e^{-x^2}$ has no vertical asymptotes.

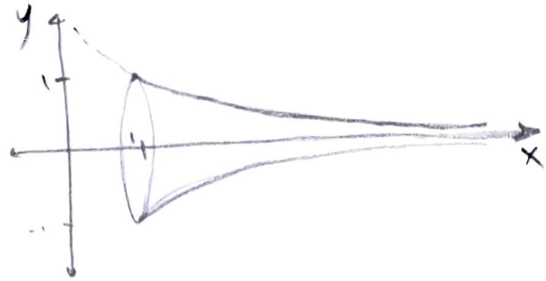
For $x \geq 1$, $0 \leq e^{-x^2} \leq xe^{-x^2}$ and

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{2} - \frac{1}{2}e^{-b^2} \to 0$$

$= \frac{1}{2}$. Convergent

Therefore $\int_1^\infty e^{-x^2}\,dx$ converges and thus the original integral converges.

2. 

$$V = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \int_1^b \frac{\pi}{x^2}\, dx = \lim_{b\to\infty} -\frac{\pi}{x}\Big|_1^b$$

$$= \lim_{b\to\infty} \pi - \frac{\pi}{b} = \pi$$

$$SA = \int_1^{\infty} 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$$

For $x \geq 1$, $0 \leq \frac{1}{x} \leq \frac{1}{x}\sqrt{1+\frac{1}{x^4}}$ (because $\sqrt{1+\frac{1}{x^4}} \geq 1$).

Also $\int_1^{\infty} \frac{1}{x}\, dx = \infty$, therefore $\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx = \infty$.

The volume is π, but the surface area is not finite.

So π gallons of paint will fill the volume, but not coat the whole surface.

Challenge. $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

a) $\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = \lim_{b\to\infty} -e^{-t}\Big|_0^b$

$= \lim_{b\to\infty} 1 - e^{-b} = \lim_{b\to\infty} 1 - \frac{1}{e^b} = 1$

Therefore $\Gamma(1) = 1$

b) $\Gamma(2) = \int_0^\infty t e^{-t}\, dt = \lim_{b\to\infty} \left[-te^{-t}\Big|_0^b + \underbrace{\int_0^b e^{-t}\, dt}_{\Gamma(1)} \right]$

IBP $u = t$ $du = dt$

$dv = e^{-t}\, dt$ $v = -e^{-t}$

$= \lim_{b\to\infty} -be^{-b} + 1 - e^{-b} = 1$

l'Hôpital's rule may help in evaluating $\lim_{b\to\infty} be^{-b}$.

c) $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t}\, dt$

IBP $u = t^x$ $du = x t^{x-1}$

$dv = e^{-t}\, dt$ $v = -e^{-t}$

$\lim_{b\to\infty} b^x e^{-b} = 0$, perhaps using l'Hôpital's rule again

Thus $\Gamma(x+1) = x \int_0^\infty t^{x-1} e^{-t}\, dt = x\,\Gamma(x)$.

Wolfram Alpha will show you a small graph: "y = Gamma(x)"