258 F20 WS-08 Series I solutions

1. $s_1 = \frac{1}{2}$ and $s_{n+1} = s_n + \frac{1}{2^{n+1}}$

$$s_2 = s_1 + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = s_2 + \frac{1}{2^3} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = s_3 + \frac{1}{2^4} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$

$$s_5 = s_4 + \frac{1}{2^5} = \frac{15}{16} + \frac{1}{32} = \frac{31}{32}$$

$$\vdots$$

$$s_n = \frac{2^n - 1}{2^n}$$

And $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1.$

this proves that $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ (since $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is defined to be $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i}$ and $\sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = s_n$).

2. $s_k = \sum_{n=1}^{k} \frac{2}{n(n+2)} = \frac{2}{3} + \frac{2}{2(4)} + \frac{2}{3(5)} + \cdots + \frac{2}{k(k+2)}$

a) $s_1 = \frac{2}{3}$

$s_2 = \frac{2}{3} + \frac{2}{2(4)} = \frac{22}{24} = \frac{11}{12}$

$s_3 = \frac{2}{3} + \frac{2}{2(4)} + \frac{2}{3(5)} = \frac{11}{12} + \frac{2}{3(5)} = \frac{165+24}{180} = \frac{189}{180}$

No obvious pattern.

b) $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$

c) $s_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots$

$\cdots + \left(\frac{1}{k-1} - \frac{1}{k+1}\right) + \left(\frac{1}{k} - \frac{1}{k+2}\right)$

All terms cancel except those circled in green.

$s_k = 1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2} = \frac{3}{2} - \frac{1}{k+1} - \frac{1}{k+2}$

d) $\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{3}{2} - \frac{1}{k+1} - \frac{1}{k+2} = \frac{3}{2}$

The point: $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{2}{n(n+1)}$ by definition

$= \frac{3}{2}$ as we calculated.

3. a) $S_1 = \frac{1}{3}$ and $S_{n+1} = S_n + \frac{1}{3^{n+1}}$

$$S_n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

and the last part of that inequality give a sequence converging to 1 (see problem 1). Seems reasonable that the sequence in this problem might also converge.

(Monotonic, bounded sequence)

b) $S_1 = 1$ and $S_{n+1} = S_n + (-1)^{n+1}$

$S_1 = 1$, $S_2 = 1 + (-1)^2 = 2$, $S_3 = 2 + (-1)^3 = 1$, $S_4 = 2$, ..

$S_5 = 1$, $S_6 = 2$, ..

The sequence alternates between 1 and 2. Not convergent.

c) $S_1 = 1$, $S_2 = 1 + \frac{1+1}{1} = 3$, $S_3 = 3 + \frac{3}{2} = \frac{9}{2}$

$S_4 = \frac{9}{2} + \frac{4}{3}$, $S_5 = S_4 + \frac{5}{4}$, $S_6 = S_5 + \frac{6}{5}$, ...

Each term adds a number greater than 1, seems like this must diverge to ∞.

3 continued.

d) $S_1 = 1$ and $S_{n+1} = S_n + \frac{1}{n+1}$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\sum_{i=1}^{\infty} \frac{1}{i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \lim_{n\to\infty} S_n = ?$$

Harmonic series.

$$S_n = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{1}{2}} + \underbrace{\cdots + \frac{1}{n}}_{\geq \frac{1}{2}}$$

No matter how far we go, the limit is always at least $\frac{1}{2}$ above ... or $\sum_{i=1}^{\infty} \frac{1}{i}$ is like adding $\frac{1}{2}$ to itself infinitely often. The series diverges to ∞.