258 F20 Solutions to WS-09 Series II 10/22/20

1. a) $S_2 = 1 + \frac{1}{4} = \frac{5}{4}$

$$\int_3^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_3^b = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{3} = \frac{1}{3}$$

$$\int_2^\infty \frac{1}{x^2}\,dx = \frac{1}{2} \quad \text{similarly}$$

By the IET: $\frac{5}{4} + \frac{1}{3} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{5}{4} + \frac{1}{2}$

width of the interval: $\frac{5}{4} + \frac{1}{2} - \left(\frac{5}{4} + \frac{1}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

b) $\int_{k+1}^\infty \frac{1}{x^2}\,dx = \frac{1}{k+1}$ and $\int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k}$

By the IET: $S_k + \frac{1}{k+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq S_k + \frac{1}{k}$

width: $S_k + \frac{1}{k} - \left(S_k + \frac{1}{k+1}\right) = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$

c) $\frac{1}{k(k+1)} \leq \frac{1}{100}$ when $k(k+1) \geq 100$. k must be an integer and thus $k \geq 10$.

d) By the IET: $S_{10} + \frac{1}{11} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq S_{10} + \frac{1}{10}$

$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.549767731$

Therefore $1.640677 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.649768$

Conclusion $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.64\ldots$ (1,6,4 are correct)

History: Finding $\sum \frac{1}{n^2}$ is the Basel problem, posed 1650 and solved 1734 by L. Euler. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.644934066685$.

2. a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+n+1}$ diverges by the test for divergence:

$$\lim_{n\to\infty} \frac{n^3}{n^3+n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}+\frac{1}{n^3}} = 1 \neq 0$$

b) $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ (both series conv, so this works)

$$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right) + \frac{2}{3}\left(\frac{1}{1-\frac{2}{3}}\right)$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

c) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges by the integral test:

$f(x) = \frac{1}{x\ln(x)}$ is positive for $x > 1$,
continuous for $x > 0$,
decreasing (since x and $\ln x$ are both increasing).

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du \text{ diverges.}$$

$u = \ln x$
$du = \frac{1}{x}dx$

3. a) $\sum_{n=1}^{\infty} \frac{1+2^n}{2+3^n}$ conv. by comparison with $\frac{1+2^n}{3^n}$

$0 \le \frac{1+2^n}{2+3^n} \le \frac{1+2^n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \frac{5}{2}$

(see 2a)

b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$ conv. by comparison with $\frac{1}{n^{3/2}}$

$0 \le \frac{1}{n\sqrt{n+1}} \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv

(p-series with $p = 3/2$)

4. a) $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n-1}$ div. by comparison with $\left(\frac{3}{2}\right)^n$

$\frac{1+3^n}{2^n-1} \ge \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n \ge 0$ and $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges

(geometric series with $r = \frac{3}{2}$)

b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln(n)}$ div. by comparison with $\frac{1}{n\ln(n)}$

$\frac{1}{\sqrt{n}\ln(n)} \ge \frac{1}{n\ln(n)} \ge 0$ for $n \ge 1$ and $\sum \frac{1}{n\ln(n)}$

diverges (see 2c).