Solutions to WS-11 Power series II : Taylor series I 11/8/20

1. $f(x) = \sin x$ $\qquad C_0 = \frac{f(\pi)}{0!} = 0$

$f'(x) = \cos x$ $\qquad C_1 = \frac{f'(\pi)}{1!} = \frac{-1}{1} = -1$

$f''(x) = -\sin x$ $\qquad C_2 = \frac{f''(\pi)}{2!} = 0$

$f^{(3)}(x) = -\cos x$ $\qquad C_3 = \frac{f^{(3)}(\pi)}{3!} = \frac{1}{3!}$

$f^{(4)}(x) = \sin x$ $\qquad C_4 = 0$

Cycles $\qquad C_5 = \frac{f^{(5)}(\pi)}{5!} = -\frac{1}{5!}$

$C_6 = 0$

$C_7 = \frac{f^{(7)}(\pi)}{7!} = \frac{1}{7!}$

$$\sin x = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!}$$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

a) $e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{(-x^2/2)^2}{2!} + \frac{(-x^2/2)^3}{3!} + \frac{(-x^2/2)^4}{4!}$

$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{2^4(4!)} - \cdots$

b) $P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$

$P_6(\frac{1}{2}) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} \approx 0.88248697917$

$e^{-\frac{(1/2)^2}{2}} = e^{-\frac{1}{8}} \approx 0.88249690258$

c) $\int_0^1 e^{-x^2/2}\, dx \approx \int_0^1 P_6(x)\, dx = \int_0^1 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\, dx$

$= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} \Big|_0^1$

$= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} \approx 0.8553571$ *

Divide by $\sqrt{2\pi}$ to get: 0.341238129129.

4. $$\frac{e^x}{1-x} = \left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)$$

$$= 1+(x+x)+\left(x^2+x^2+\frac{x^2}{2}\right)+\left(x^3+x^3+\frac{x^3}{2}+\frac{x^3}{6}\right)+\cdots$$

$$= 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots$$

* In 3c $1-\frac{1}{6}+\frac{1}{40}-\frac{1}{336}$ is within $\frac{1}{2^4(4!)9}$ of the true value of $\int_0^1 e^{-x^2/2}\,dx$, since we have an alternating series and $\frac{1}{2^4(4!)9}$ is the next term. $\frac{1}{2^4(4!)9} = \frac{1}{3456}$, so we're pretty close.