POWER SERIES II: TAYLOR SERIES I

Definition. Let f be a function with derivatives of all orders near the number a. The **Taylor series generated by f at x = a** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

The degree n Taylor polynomial generated by f at $x = a$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

1. Find the Taylor series for $\sin(x)$ at $x = \pi$.

2. Yesterday we found that the Taylor series generated by $\sin(x)$ at $x = 0$ is

$$\sum_{k=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Take the derivative to find a power series for $\cos(x)$.

3. This problem concerns the function $f(x) = e^{-\frac{x^2}{2}}$, which is the heart of the standard normal distribution. Integrating this function is essential for most statistics, but exact integrals generally are impossible to calculate. Instead, we'll use Taylor polynomials to find approximations. Recall that the Taylor series for e^x centered at 0 is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

a) Use the Taylor series for e^x to find a power series for $f(x) = e^{-\frac{x^2}{2}}$. Write down the first 5 terms (there's no need to describe the pattern in sigma notation).

b) Use the degree 6 polynomial at the beginning of your series to estimate $f(1/2)$ (use a calculator and compare your estimate with the real value).

c) Use the degree 6 polynomial at the beginning of your series to estimate $\displaystyle\int_0^1 e^{-\frac{x^2}{2}}\,dx$. According to statistical tables, dividing your answer by $\sqrt{2\pi}$ should give a number close to 0.34134.

4. Multiply the power series for e^x and $\frac{1}{1-x}$ together to find a power series for $\dfrac{e^x}{1-x}$. Write down the first 4 terms of the result (through the x^3 term).

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