FUNCTIONS DEFINED AS INTEGRALS

Definition. The natural logarithm function is defined as the following integral:

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt}$$

See the Desmos graph: `https://www.desmos.com/calculator/vmhz3j2hsk`.

1. All of the properties of the natural logarithm follow from the definition above. Use the definition to prove part a, then use part a to prove part b, and so on.

a) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

b) $\ln(ax) = \ln a + \ln x$

c) $\ln\left(\dfrac{x}{a}\right) = \ln x - \ln a$

d) $\ln\left(x^a\right) = a \ln x$

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Definition. The natural exponential function is the inverse of the natural logarithm function:

$$e^x = y \text{ if and only if } x = \ln y$$

or, equivalently,

$$e^{\ln x} = x \text{ and } \ln\left(e^x\right) = x$$

See the Desmos graph: `https://www.desmos.com/calculator/vmhz3j2hsk`.

2. The properties of the natural exponential function follow from the properties of the natural logarithm.

a) Use implicit differentiation of $x = \ln y$ to find $\frac{dy}{dx}$ for $y = e^x$.
b) Take the logarithm of $e^a e^x$ to show that $e^{a+x} = e^a e^x$.
c) Take the logarithm to show $e^{x-a} = e^{-a} e^x$.
d) Take the logarithm to show $\left(e^x\right)^a = e^{ax}$.

Definition. The general exponential function with base b is $b^x = e^{x \ln b}$ and the logarithm with base b is $\log_b x = \dfrac{\ln x}{\ln b}$.

3. Show that the exponential function with base b and logarithm with base b are inverses, that is that $x = b^{\log_b x}$ and $x = \log_b\left(b^x\right)$.

Definition. The **gamma function** is defined as $\boxed{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt}$ for $\alpha > 0$.

4. The goal of this problem is to investigate the gamma function and its connection to the factorial.

a) Show that $\Gamma(1) = 1$ by evaluating the integral.
b) Integrate by parts to find $\Gamma(2)$. Integration by parts: $\int udv = uv - \int vdu$. Hint: watch for $\Gamma(1)$ to show up–you don't need to repeat your work for this integral.
c) Integrate by parts to show that $\Gamma(3) = 2\Gamma(2)$.
d) Fill in the end of the statement: for any positive integer n, $\Gamma(n) =$

The gamma function is a way to extend the factorials to non-integers (and ultimately even to complex numbers). For example $\Gamma(1/2) = \sqrt{\pi}$. For more, see `https://mathworld.wolfram.com/GammaFunction.html`.