EXPONENTIAL GROWTH AND HYPERBOLIC FUNCTIONS

Example. The SIR model of disease spread looks at the interaction of 3 quantities: at time t $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infected individuals, and $R(t)$ is the number of recovered (or removed) individuals. The model starts with assumptions about the rates of change of these variables:

$$\frac{dR}{dt} = kI$$
$$\frac{dS}{dt} = -b\left(\frac{S}{p_0}\right)I$$
$$\frac{dI}{dt} = b\left(\frac{S}{p_0}\right)I - kI$$

where k is a recovery rate constant and b is a contagion rate constant and p_0 is the population size (also constant). The full model is a bit too complex for us, so we'll just look at a simple model for $I(t)$ near time $t = 0$ when $S \approx p_0$. This gives a simplified picture that should be reasonably accurate for initial spread of the disease.

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1. Population tend to grow in proportion to their size: $P'(t) = rP(t)$ where r is a constant growth rate parameter. This means $P(t) = p_0 e^{rt}$ where p_0 is the initial population.

a) Human population is currently about 7.8 billion and the estimated growth rate is about 1% per year. This means that $p_0 = 7.8$ billion and $P(1) = 1.01p_0$. Solve this equation for r.
b) Predict the population in 10 years and in 100 years.
c) What does the model predict in the long-term? Is this realistic?
d) $r = b - d$ where b is a birth rate parameter and d is a death rate parameter. What has to be true of b and d if the population isn't growing exponentially?

Challenge. The exponential model for population growth $P' = rP$ predicts unbounded population sizes. The logistic model resolves this problem: $P' = aP(1-bP)$ where a and b are positive constants. If it helps you to solve the problem, you may use $a = 1$ and $b = 1/2$.

a) Solve the integral equation for $P(t)$ (you may need to use the method of partial fractions):

$$\int \frac{1}{P(1-bP)}\, dP = \int a\, dt$$

b) Calculate $\lim_{t\to\infty} P(t)$. Does the logistic model predict unbounded growth?

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $T' = -k(T - T_m)$ where k is a positive constant of proportionality and T_m is the (constant) temperature of the environment. This means that $T = T_m + T_0 e^{-kt}$ where $T_0 = T(0) - T_m$.

2 (CSI Gonzaga)**.** A cooling cup of coffee is found outside on a 5°C day. At 12:15 its temperature is 35°C and at 12:45 its temperature is 25°C.

a) Use the two points to find a formula for $T(t)$, the temperature of the coffee t hours after 12:15 (so $t = 0$ is 12:15).
b) Coffee is usually brewed at about 95°C. Use this to estimate how long ago the coffee was brewed.

Definition. The hyperbolic sine and hyperbolic cosine functions are:

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic identities, derivatives, and integrals: `https://web02.gonzaga.edu/faculty/axon/258/P446.pdf`. They also have inverse functions, possibly mostly useful as antiderivatives: `https://web02.gonzaga.edu/faculty/axon/258/P450.pdf`.