1. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

Solution. $\frac{52}{3}$

2. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

Solution. $2(\pi + e - \frac{1}{e})$

3. Find the volume of the solid under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

Solution. $\int_{-2}^{2} \int_{y^2}^{4} 1 + x^2y^2 \; dxdy$ or $\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 1 + x^2y^2 \; dydx$ which give $\frac{2336}{27}$ (numbers on the test will be nicer).

4. Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

Solution. $\int_0^2 \int_0^{2y} \sqrt{4 - y^2} \; dxdx$ or $\int_0^{\frac{\pi}{2}} \int_0^2 2r^2 \cos\theta \; drd\theta$ which give $\frac{16}{3}$.

5. Reverse the order of integration for the integral $\int_0^2 \int_{x^2}^4 f(x, y) \; dydx$.

Solution. $\int_0^4 \int_0^{\sqrt{y}} f(x, y) \; dxdy$

6. Evaluate the integral by reversing the order of integration $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2) \; dxdy$.

Solution. $\int_0^{\sqrt{\pi}} \int_0^x \cos(x^2) \; dydx = 0$

7. Find the area of the polar region $R = \{(r, \theta) \mid 0 \le r \le 2\sin\theta, \; \frac{\pi}{2} \le \theta \le \pi\}$.

Solution. $A = \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\sin\theta} r \; drd\theta = \frac{\pi}{2}$

8. Evaluate the integral $\iint_R \sin(x^2 + y^2) \; dA$ where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

Solution. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 r\sin(r^2) \; drd\theta = \frac{\pi}{8}(1 - \cos 4)$

9. Determine the volume of the solid below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.

Solution. $\int_0^{2\pi} \int_0^3 (18 - 2r^2) r drd\theta = 81\pi$

10. Find the center of mass of the lamina that occupies the region enclosed by the lines $x = 0$, $y = x$, and $2x + y = 6$ and has density function $\rho(x, y) = x^2$.

Solution. $m = \int_0^2 \int_x^{6-2x} x^2 \; dydx = 4$, $M_y = \int_0^2 \int_x^{6-2x} x^3 \; dydx = \frac{24}{5}$, and $M_x = \int_0^2 \int_x^{6-2x} yx^2 \; dydx = \frac{48}{5}$. Therefore $(\overline{x}, \overline{y}) = (\frac{6}{5}, \frac{12}{5})$.

11. A lamina occupies the region inside the circle $x^2 + y^2 = 2x$ but outside the circle $x^2 + y^2 = 1$ and has density $\rho(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. Find the center of mass of the lamina.

Solution. $m = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} dr d\theta = 2\sqrt{3} - \frac{2\pi}{3}$, $M_y = \sqrt{3}$, and $M_x = 0$ (also apparent from symmetry). Therefore $(\overline{x}, \overline{y}) = (\frac{3\sqrt{3}}{6\sqrt{3}-2\pi}, 0)$.

12. Evaluate the integral $\iiint_E \sin y \; dV$ where E is the region below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$.

Solution. $\int_0^\pi \int_0^{\pi-x} \int_0^x \sin y \; dz dy dx$

13. Find the center of mass of the solid bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$ and with density function $\rho(x, y, z) = 4$.

Solution. $m = \int_0^1 \int_0^{1-z} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} 4 \; dy dx dz = \frac{16}{5}$, $M_{yz} = \frac{8}{7}$, $M_{xz} = 0$, and $M_{xy} = \frac{32}{35}$. Therefore $(\overline{x}, \overline{y}, \overline{z}) = (\frac{5}{14}, 0, \frac{2}{7})$.

14. Find the moments of inertia of a rectangular brick with dimensions a, b, and c, mass M, and constant density if the center of the brick is at the origin and the edges are parallel to the coordinate axes.

Solution. It turns out I skipped this in class, so it won't be on the exam. If you're interested, the discussion is on page 733.

15. Evaluate $\iiint_E x \; dV$ where E is the solid enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution. $\int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 5} r^2 \cos\theta \; dz dr d\theta = \frac{65\pi}{4}$

16. Evaluate $\iiint_H 9 - x^2 - y^2 \; dV$ where H is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

Solution. $\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 9\rho^2 \sin\phi - \rho^4 \sin^3\phi \; d\rho d\theta d\phi = \frac{486\pi}{5}$

17. Find the centroid of the solid (with uniform density 1) inside $x^2 + y^2 + z^2 = 4$, outside $x^2 + y^2 + z^2 = 1$ and above $z = 0$.

Solution. $m = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_1^2 \rho^2 \sin\phi \; d\rho d\theta d\phi = \frac{14\pi}{3}$, $M_{yz} = 0$, $M_{xz} = 0$, $M_{xz} = \frac{15\pi}{4}$. Therefore $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, \frac{45}{56})$.

18. Find the Jacobian of the transformation $x = v$, $y = u(1 + v^2)$.

Solution. $\frac{\partial(x,y)}{\partial(u,v)} = -(1 + v^2)$

19. Use the transformation $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$ to evaluate the integral $\iint_R 4x + 8y \; dA$ where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$.

Solution. $\int_0^8 \int_{-4}^4 [u + v + 2(v - 3u)] \, \frac{1}{4} \; du dv = 192$