NAME(S): Solutions

MATH 259 MOTION IN SPACE MARCH 4, 2014

1. A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. Our goal is to find $\mathbf{r}(t)$, the position of the object at time t. To simplify matters we assume that air resistance is negligible and that gravity is the only force acting on the projectile.

a) Find the acceleration of the projectile $\mathbf{a}(t)$ (this is a vector).

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

b) Antidifferentiate $\mathbf{a}(t)$ and use the initial condition $\mathbf{v}(0) = \mathbf{v}_0$ to find the velocity $\mathbf{v}(t)$.

$$\vec{v}(t) = \langle 0, -9.8 \rangle t + \vec{C}$$

$$\vec{v}_0 = \vec{v}(0) = \vec{C}$$

$$\vec{v}(t) = \langle 0, -9.8 \rangle t + \vec{v}_0$$

c) Antidifferentiate again to find the position $\mathbf{r}(t)$.

$$\vec{r}(t) = \langle 0, -9.8 \rangle \frac{t^2}{2} + \vec{v}_0 t + \vec{C}$$

$\vec{r}(0) = \vec{C}$ initial position

d) Express $\mathbf{v}_0$ as a function of α and initial speed v_0 and substitute this into your formula for $\mathbf{r}(t)$.

$$\vec{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$$

$$\vec{r}(t) = \langle 0, -9.8 \rangle \frac{t^2}{2} + \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle t + \vec{C}$$

$$= \langle (v_0 \cos\alpha)t + C_1, -4.9t^2 + (v_0 \sin\alpha)t + C_2 \rangle$$

2. A projectile is fired from a height of 2 m with an initial velocity of 80 m/s at an angle of 30° above horizontal. Determine how far the projectile travels horizontally before it hits the ground ($y = 0$).

Initial position $\vec{c} = \langle 0, 2 \rangle$

Initial velocity $\vec{v}_0 = \langle 80\cos(\frac{\pi}{6}), 80\sin(\frac{\pi}{6}) \rangle = \langle 40\sqrt{3}, 40 \rangle$

$\vec{r}(t) = \langle 40\sqrt{3}\,t, -4.9t^2 + 40t + 2 \rangle$

Time of flight: $0 = -4.9t^2 + 40t + 2$

$$t = \frac{-40 \pm \sqrt{40^2 - 4(-4.9)2}}{-9.8} = \frac{-40 \pm \sqrt{1600 + 39.2}}{-9.8}$$

$$\approx 8.213 \quad \text{or} \quad -.050$$

we're interested in $t = 8.213$ s:

$\vec{r}(8.213) = 40(\sqrt{3})8.213 \approx \langle 569.01, 0 \rangle$

The projectile travels 569.01 meters.

3. It is often convenient to decompose the acceleration of an object into two components: one in the direction of the tangent (**T**) and one in the direction of the normal (**N**). Recall that

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{\mathbf{v}}{v} \tag{1}$$

and

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} \tag{2}$$

a) Solve equation 1 for **v** and differentiate both sides to get a formula for **a**.

$\vec{v} = v\vec{T}, \quad \vec{a} = \vec{v}' = v'\vec{T} + v\vec{T}'$

b) Solve equation 2 for $\mathbf{T}'(t)$ and substitute into your answer for part a. You should now have an expression of $\mathbf{a}(t)$ as the sum of scalar multiples of **T** and **N**.

$\vec{T}' = |\vec{T}'|\vec{N}. \qquad \vec{a} = v'\vec{T} + v|\vec{T}'|\vec{N}$

c) It is customary to make the substitution $|\mathbf{T}'| = \kappa v$ where κ is the curvature. Check that this gives you $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.

$\vec{a} = v'\vec{T} + v\kappa v\vec{N} = v'\vec{T} + \kappa v^2\vec{N}$

4. The magnitude of the tangential component of acceleration is $a_T = v'$ and the magnitude of the normal component of acceleration is $a_N = \kappa v^2$.

a) Explain (in English) what happens when $a_T = 0$.

v is the speed. $0 = a_T = v'$ means that the speed is constant. The direction of travel may change, but the speed of travel does not.

b) Explain (in English) what happens when $a_N = 0$.

$a_N = 0$, means that the object is either stationary or moving in a straight line (at any speed).

c) What is the significance of the absence of the binormal vector **B** in our decomposition of acceleration?

All acceleration happens in the plane normal to $\vec{B}$. Acceleration is the result of force, hence $\vec{B}$ is always perpendicular to the force acting on the object.

5. Consider an object with position $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ (the twisted cubic).

a) Find the tangential and normal components of acceleration (you are welcome to cite old work here).

$\vec{v}(t) = \langle 1, 2t, 3t^2 \rangle \qquad v(t) = |\vec{v}(t)| = \sqrt{1+4t^2+9t^4}$

$\vec{a}(t) = \langle 0, 2, 6t \rangle \qquad v'(t) = \frac{1}{2}(1+4t^2+9t^4)^{-\frac{1}{2}}(8t+36t^3)$

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{\sqrt{4+36t^2+36t^4}}{(1+4t^2+9t^4)^{3/2}}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 12t^2 - 6t^2, -6t, 2 \rangle = \langle 6t^2, -6t, 2 \rangle$$

$$a_T = v' = \frac{1}{2}(1+4t^2+9t^4)^{-\frac{1}{2}}(8t+36t^3)$$

$$a_N = \kappa v^2 = \frac{\sqrt{4+36t^2+36t^4}}{(1+4t^2+9t^4)^{3/2}}(1+4t^2+9t^4) = \sqrt{\frac{4+36t^2+36t^4}{1+4t^2+9t^4}}$$

b) Find the tangential and normal components of acceleration at the points $(0,0,0)$ and $(1,1,1)$.

$(0,0,0)$ corresponds to $t=0$:

$$a_T(0) = \frac{1}{2}(1)^{-\frac{1}{2}}0 = 0$$

$$a_N(0) = \sqrt{\frac{4+0}{1+0}} = 2$$

$(1,1,1)$ corresponds to $t=1$:

$$a_T(1) = \frac{1}{2}(1+4+9)^{-\frac{1}{2}}(8+36) = \frac{1}{2}\left(\frac{1}{\sqrt{14}}\right)44 = \frac{22}{\sqrt{14}}$$

$$a_N(1) = \sqrt{\frac{4+36+36}{1+4+9}} = \sqrt{\frac{76}{14}} = \sqrt{\frac{38}{7}}$$