MATH 259 CHAPTER 9 EXAM JANUARY 31, 2014

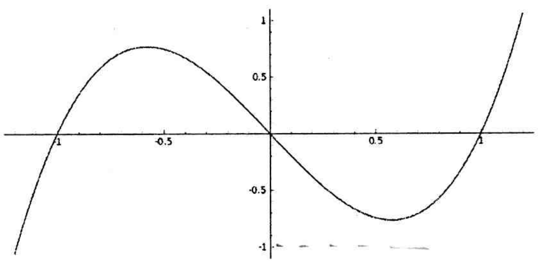
INSTRUCTIONS: Answer all 11 problems. Show all your work: even correct answers may receive little or no credit if a method of solution is not shown. Calculators, notes, cell phones, and other materials are not permitted.

NAME. Solutions

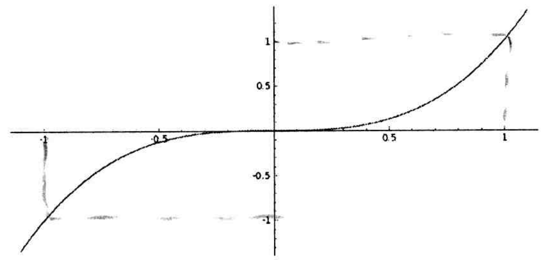
You may find the following helpful:

- Half-angle formulas: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$;
- Derivative of a polar curve: $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$;
- Arc length of a polar curve: $L = \displaystyle\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$;
- Conic sections with foci at the origin and directrices parallel to an axis: $r = \dfrac{ed}{1 \pm e\cos\theta}$ or $r = \dfrac{ed}{1 \pm e\sin\theta}$;
- Some values of $\tan\theta$: $\tan 0 = 0$, $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$, $\tan(\frac{\pi}{4}) = 1$, $\tan(\frac{\pi}{3}) = \sqrt{3}$, and $\tan(\frac{\pi}{2})$ is undefined.

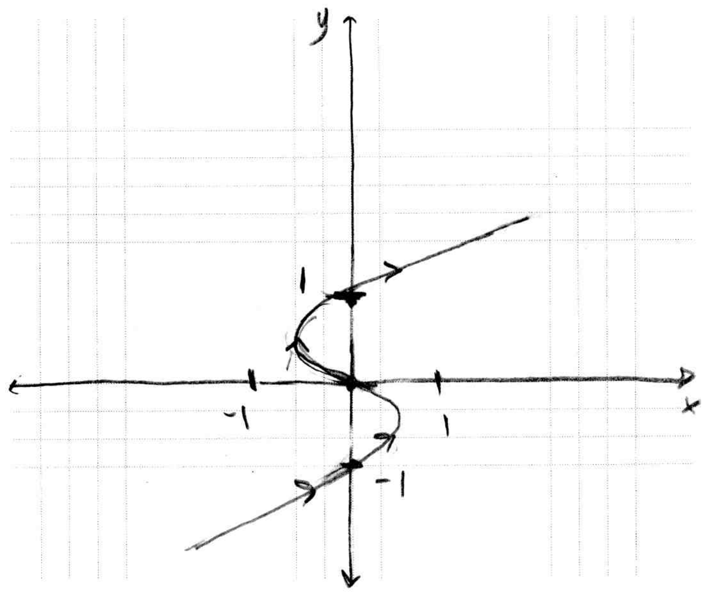
1. Use the following graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate the direction in which the curve is traced as t increases and give coordinates for axis intercepts.



$x = f(t)$



$y = g(t)$



2. Eliminate the parameter to find a Cartesian equation for the curve with parametric equations $x = t^3$, $y = 2t - 1$.

Option 1: $\sqrt[3]{x} = t$ $\boxed{y = 2\sqrt[3]{x} - 1}$

Option 2: $\frac{y+1}{2} = t$ $\boxed{x = \left(\frac{y+1}{2}\right)^3}$

Either is correct.

3. Find the slopes of the two lines tangent to the parametric curve $x(t) = t - t^{-1}$, $y(t) = 1 + t^2$ at the point $(x, y) = (0, 2)$.

$\frac{dx}{dt} = 1 + \frac{1}{t^2}$ $\frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{2t}{1 + 1/t^2}$$

$0 = x(t) = t - t^{-1} \Rightarrow t^2 - 1 = 0 \Rightarrow (t-1)(t+1) = 0$

$2 = 1 + t^2 \Rightarrow t^2 - 1 = 0 \Rightarrow (t-1)(t+1) = 0$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{1+\frac{1}{1}} = 1$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{-2}{1+1} = -1$$

4. Find $\frac{d^2y}{dx^2}$ for $x(t) = \sin t$, $y(t) = \cos t$.

$\frac{dy}{dt} = -\sin t \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$

$\frac{d}{dt}\left[\frac{dy}{dx}\right] = -\sec^2 t$

$\frac{d^2y}{dx^2} = \frac{-\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t} = -\sec^3 t$

5. Find the area enclosed between the parametric curve $x = 1 + t^2$, $y = t - t^2$ and the x-axis. Hint: the x-axis is the line $y = 0$.

$0 = y = t - t^2 = t(1-t) \Rightarrow t = 0,\ t = 1$

$\frac{dx}{dt} = 2t$

$\text{Area} = \int_0^1 (t - t^2)(2t)\,dt = \int_0^1 2t^2 - 2t^3\,dt = \frac{2}{3}t^3 - \frac{1}{2}t^4 \Big|_0^1$

$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

6. Convert the Cartesian coordinates to polar coordinates.

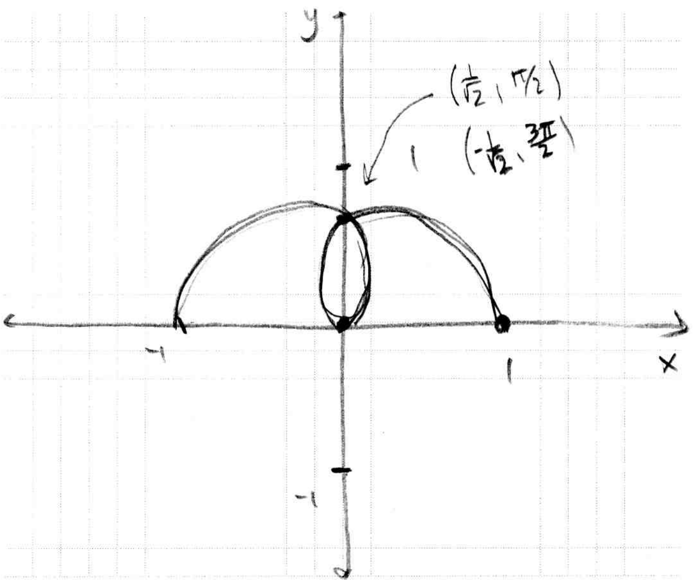
a) $(x, y) = (0, 2)$ $\quad (r, \theta) = (2, \pi/2)$

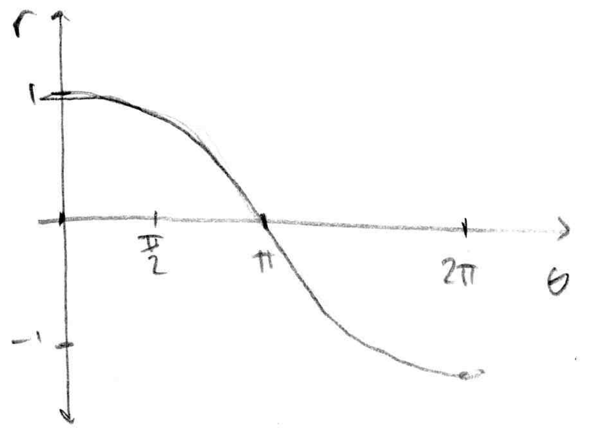
b) $(x, y) = (-1, 1)$ $\quad (r, \theta) = (\sqrt{2}, \frac{3\pi}{4})$

c) $(x, y) = (1, -\sqrt{3})$ $\quad (r, \theta) = (2, \frac{-\pi}{3})$

All of these have other correct answers.

7. Sketch the polar curve $r = \cos\left(\frac{\theta}{2}\right)$ for $0 \le \theta \le 2\pi$. Give coordinates for axis intercepts.





8. Find the slope of the line tangent to the polar curve $r = 2 + \sin(3\theta)$ when $\theta = \frac{\pi}{3}$.

$x = (2+\sin 3\theta)\cos\theta$ $\qquad \frac{dx}{d\theta} = 3\cos 3\theta \cos\theta - \sin\theta(2+\sin 3\theta)$

$y = (2+\sin 3\theta)\sin\theta$ $\qquad \frac{dy}{d\theta} = 3\cos 3\theta \sin\theta + \cos\theta(2+\sin 3\theta)$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{3\cos\pi\sin\pi/3 + \cos\pi/3\,(2+\overset{0}{\cancel{\sin\pi}})}{3\cos\pi\cos\pi/3 - \sin\pi/3\,(2+\underset{0}{\cancel{\sin\pi}})}$$

$$= \frac{3(-1)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}(2)}{3(-1)\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2}(2)} = \frac{2-3\sqrt{3}}{-3-2\sqrt{3}} = \frac{3\sqrt{3}-2}{2\sqrt{3}+3}$$

9. Determine the area of one loop of the polar curve $r = \sin 3\theta$.

$$\int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\, d\theta = \int_0^{\pi/3} \frac{1}{4}(1-\cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/3} 1-\cos 6\theta\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$$

$$= \frac{\pi}{12}$$

10. Find an integral giving the arc length of the polar curve $r = \cos\left(\frac{\theta}{2}\right)$. There is no need to evaluate your integral.

$$2 \int_0^{2\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

or

$$\int_0^{4\pi} \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

11. Determine if the conic section defined by $r = \dfrac{6}{4 - 2\sin\theta}$ is an ellipse, a parabola, or a hyperbola and find a Cartesian equation for its directrix.

$$r = \frac{6/4}{1 - 2/4 \sin\theta}$$

$$r = \frac{3/2}{1 - 1/2 \sin\theta}$$

$e = \frac{1}{2}$

$d = 3$

Directrix $y = -3$, ellipse.