NAME(S): Solutions

MATH 259 MOMENTS APRIL 13, 2015

INSTRUCTIONS: Set up the integrals for all problems before attempting to evaluate any of them.

Definiton. The mass of a lamina with density $\rho(x,y)$ occupying region D in the xy-plane is

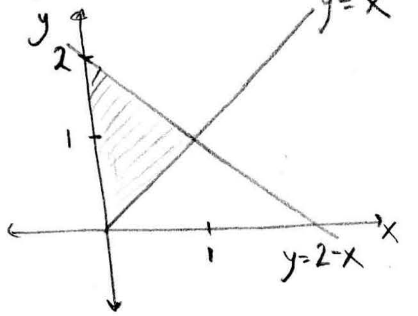
$$m = \iint_D \rho(x,y)\, dA.$$

The moments about the x-axis and y-axis of the same lamina are (respectively)

$$M_x = \iint_D y\rho(x,y)\, dA \text{ and } M_y = \iint_D x\rho(x,y)\, dA.$$

The center of mass of the lamina is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{M_y}{m}$ and $\overline{y} = \frac{M_x}{m}$.

1. Find the center of mass of a lamina with density $\rho(x,y) = y$ occupying the triangular region bounded by $x = 0$, $y = x$, and $y = 2 - x$.



$$m = \int_0^1 \int_x^{2-x} y \, dy\, dx = \int_0^1 \tfrac{1}{2}(2-x)^2 - \tfrac{1}{2}x^2 \, dx = \int_0^1 \tfrac{1}{2}(4-4x+x^2) - \tfrac{1}{2}x^2 \, dx$$

$$= \int_0^1 2-2x \, dx = 2x - x^2 \Big|_0^1 = 2-1 = 1$$

$$M_x = \int_0^1 \int_x^{2-x} y^2 \, dy\, dx = \ldots =$$

$$M_y = \int_0^1 \int_x^{2-x} xy \, dy\, dx = \ldots =$$

$$\bar{x} = \frac{M_y}{m} = \qquad \bar{y} = \frac{M_x}{m} =$$

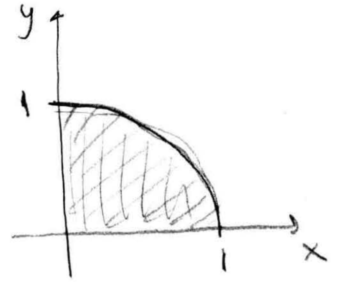
Definiton. The moment of inertia (also called the second moment) of a particle of mass m at distance r from the axis of rotation is mr^2. For a lamina with density $\rho(x,y)$ occupying region D in the xy-plane, the moments of inertia about the x-axis and y-axis are (respectively)

$$I_x = \iint_D y^2\rho(x,y)\, dA \text{ and } I_y = \iint_D x^2\rho(x,y)\, dA.$$

The moment of inertia about the origin is

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA.$$

2. Find the moments of inertia (I_x, I_y, and I_0) of a lamina with density $\rho(x,y) = x$ and occupying $D = \{(x,y) \mid x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$.



$$I_x = \int_0^{\frac{\pi}{2}}\int_0^1 (r\sin\theta)^2\, r\cos\theta\; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r^4 \sin^2\theta\cos\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5}\sin^2\theta\cos\theta\,d\theta = \frac{1}{15}\sin^3\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{15}$$

$$I_y = \int_0^{\frac{\pi}{2}}\int_0^1 (r\cos\theta)^2\, r\cos\theta\; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos^3\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5}\cos\theta(1-\sin^2\theta)\,dr\,d\theta = \frac{1}{5}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{1}{5}\left(1-\frac{1}{3}\right) = \frac{2}{15}$$

$$I_0 = \int_0^{\frac{\pi}{2}}\int_0^1 (r^2)\,r\cos\theta\; r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{5}\cos\theta\,d\theta = \frac{1}{5}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{5}$$

Note that $I_0 = I_x + I_y$ by definition. Use this as a way of checking the answers above: $\frac{1}{5} = \frac{1}{15} + \frac{2}{15}$ ✓

3. Find the center of mass of the solid of uniform density occupying the region below the surface $z = 1 - x^2 - y^2$, above the xy-plane, and inside the circle $x^2 + y^2 = x$.

$$m = \int_0^{\pi}\int_0^{\cos\theta} (1-r^2)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{\cos\theta} r - r^3\,dr\,d\theta = \int_0^{\pi} \frac{1}{2}\cos^2\theta - \frac{1}{4}\cos^4\theta\,d\theta$$

$$= \cdots = \frac{5\pi}{32}$$

$$M_x = 0$$

$$M_y = \int_0^{\pi}\int_0^{\cos\theta} (1-r^2)\,r\cos\theta\; r\,dr\,d\theta = \int_0^{\pi}\int_0^{\cos\theta} r^2\cos\theta - r^4\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{3}\cos^4\theta - \frac{1}{5}\cos^6\theta\,d\theta = \cdots = \frac{\pi}{16}$$

$$\bar{x} = \frac{\pi}{16}\left(\frac{32}{5\pi}\right) = \frac{2}{5}$$

$$\bar{y} = 0$$