LINE INTEGRALS

1. Let C be the straight line from $(1,0,0)$ to $(0,1,1)$.

a) Parameterize the curve as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

b) Find $ds = |\mathbf{r}'(t)|\, dt$.

c) Evaluate the scalar line integral: $\displaystyle\int_C xy + 2z \; ds$

Method. Suppose a wire is bent in the shape of a curve C and the wire has a linear density of $\rho(x,y)$. The mass of the wire is $m = \int_C \rho(x,y)\; ds$ and the center of mass of the wire is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\; ds \quad \text{and} \quad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\; ds$$

2. Suppose a wire of constant density is bent into the shape of a semicircle of radius 2 centered at the origin. Find the center of mass of the wire.

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Definition. Let $\mathbf{F}$ be a continuous vector field and let C be a closed smooth oriented curve. The **circulation** of F on C is $\int_C \mathbf{F} \cdot d\mathbf{r}$. (A closed curve is one that follows a loop; $d\mathbf{r} = \mathbf{r}'(t)dt$).

3. Investigate the circulation of different vector fields on the unit circle C. The parameterization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ gives the circle a counterclockwise orientation. Plot the circle in CalcPlot3D. Add the the vector field to the plot for each part of the problem. Calculate the circulation for each vector field and try to figure out why it's called circulation.

a) $\mathbf{F} = \langle x, y \rangle$

b) $\mathbf{F} = \langle -y, x \rangle$

c) $\mathbf{F} = \langle y, -x \rangle$

Theorem. *Let C be a smooth curve parameterized by $\mathbf{r}(t)$, $a \le t \le b$, and let $\mathbf{F}$ be a conservative vector field with potential function φ (this means $\nabla\varphi = \mathbf{F}$). Then* $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a))$.

4. In our last example yesterday we worked with the vector field $\mathbf{F}(x, y) = \langle -x, -y \rangle$ and the curves $C_1 : \mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$ and $C_2 : \mathbf{r}_2(t) = \langle 1 - t, t \rangle$, $0 \le t \le 1$. These curves both start at $(1, 0)$ and end at $(0, 1)$ and we found that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Use the theorem above to show that the line integral of this vector field is the same for any two curves that start and end at the same points.