MOTION IN SPACE

1. Suppose the position of an object at time t is given by $\mathbf{r}(t) = \langle 2e^{2t}+1, e^{2t}-1, 2e^{2t}-10 \rangle$.

a) Find the velocity (a vector) and speed (a scalar) of the object when (both are functions of t).

$\vec{v}(t) = \langle 4e^{2t}, 2e^{2t}, 4e^{2t} \rangle$

$s(t) = \sqrt{16e^{4t} + 4e^{4t} + 16e^{4t}} = \sqrt{36e^{4t}} = 6e^{2t}$

b) Find the acceleration of the object (a vector, also a function of t).

$\vec{a}(t) = \langle 8e^{2t}, 4e^{2t}, 8e^{2t} \rangle$

2. A trajectory $\mathbf{r}(t)$ lies on the surface of a sphere of radius c centered at the origin if $|\mathbf{r}(t)| = c$. Determine if the trajectory $\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t \rangle$ lies on the surface of a sphere and evaluate $\mathbf{r}(t) \cdot \mathbf{v}(t)$. What can you conclude about the angle between $\mathbf{r}(t)$ and $\mathbf{v}(t)$?

$|\vec{r}(t)| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} = \sqrt{25(\sin^2 t + \cos^2 t)} = 5$

The trajectory lies on the surface of a sphere of radius 5.

$\vec{v}(t) = \langle 3\cos t, -5\sin t, 4\cos t \rangle$

$\vec{r}(t) \cdot \vec{v}(t) = 9\sin t\cos t - 25\cos t\sin t + 16\sin t\cos t = 0$

Therefore $\vec{r}(t)$ and $\vec{v}(t)$ are always orthogonal (they meet at a right angle).

Date: September 11, 2019.

3. In this problem you'll show that your conclusion about the angle between $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in the previous problem is part of a larger pattern.

a) Use the dot product rule for derivatives to evaluate $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)]$ and express the answer in terms of $\mathbf{r}(t) \cdot \mathbf{v}(t)$.

$$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t)$$
$$= 2\left[\vec{v}(t)\cdot\vec{r}(t)\right]$$

b) Verify that for any vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

$$\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + u_3^2 = \left(\sqrt{u_1^2+u_2^2+u_3^2}\right)^2 = |\vec{u}|^2$$

c) Suppose that the trajectory $\mathbf{r}(t)$ lies on the surface of a sphere of radius c centered at the origin. What does this mean about $\mathbf{r}(t)\cdot\mathbf{r}(t)$? What does this tell you about $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)]$?

Because $\vec{r}(t)$ lies on the surface of a sphere of radius c, we know that $|\vec{r}(t)| = c$. Hence $\vec{r}(t)\cdot\vec{r}(t) = c^2$; therefore $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \frac{d}{dt}(c^2) = 0$.

d) Show that $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

From part a and c $0 = \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = 2[\vec{v}(t)\cdot\vec{r}(t)]$.

Thus $0 = \vec{v}(t)\cdot\vec{r}(t)$. Therefore $\vec{v}(t)$ and $\vec{r}(t)$ are orthogonal.

e) Explain what this means in (relatively) normal English.

Any object moving on the surface of a sphere (e.g. a human on planet earth) has a velocity perpendicular to its position (as measured from the center of the sphere).