LINE INTEGRALS

1. Let C be the straight line from $(1,0,0)$ to $(0,1,1)$.

a) Parameterize the curve C : $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$

b) Calculate $|\mathbf{r}'(t)|$

c) Evaluate the scalar line integral: $\displaystyle\int_C xy + 2z\ ds$

Method. Suppose a wire is bent in the shape of a curve C and the wire has a linear density of $\rho(x, y)$. The mass of the wire is $m = \int_C \rho(x, y)\ ds$ and the center of mass of the wire is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\ ds \quad \text{and} \quad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\ ds$$

2. Suppose a wire of constant density is bent into the shape $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin r\rangle$ with $0 \leq t \leq \pi$ (a semicircle of radius 2 centered at the origin). Find the center of mass of the wire. Hint: use symmetry to find $\overline{x}$.

Date: November 13, 2019.

Definition. Let $\mathbf{F}$ be a continuous vector field and let C be a closed smooth oriented curve. The **circulation** of F on C is $\int_C \mathbf{F} \cdot d\mathbf{r}$. (A closed curve is one that follows a loop; $d\mathbf{r} = \mathbf{r}'(t)dt$).

3. Investigate the circulation of different vector fields on the unit circle C. The parameterization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ gives the circle a counterclockwise orientation. Plot the circle in CalcPlot3D. Add the the vector field to the plot for each part of the problem. Calculate the circulation for each vector field and try to figure out why it's called circulation.

a) $\mathbf{F} = \langle x, y \rangle$

b) $\mathbf{F} = \langle -y, x \rangle$

c) $\mathbf{F} = \langle y, -x \rangle$

Theorem. *Let C be a smooth curve parameterized by $\mathbf{r}(t)$, $a \le t \le b$, and let $\mathbf{F}$ be a conservative vector field with potential function φ (this means $\nabla \varphi = \mathbf{F}$). Then* $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a))$.

4. In one of our examples we worked with the vector field $\mathbf{F}(x, y) = \langle -x, -y \rangle$ and the curves $C_1 : \mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$ and $C_2 : \mathbf{r}_2(t) = \langle 1-t, t \rangle$, $0 \le t \le 1$. We found that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. These curves both start at $(1, 0)$ and end at $(0, 1)$, so the theorem above might explain why these two integrals are equal.

a) Find a potential function for the vector field $\mathbf{F}(x, y) = \langle -x, -y \rangle$.

b) Use the theorem (and the potential function) to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ for any curve C that starts at $(1, 0)$ and ends at $(0, 1)$