SURFACE INTEGRALS AND STOKES' THEOREM

Definition. The **surface integral** of the scalar-valued function $f(x, y, z)$ over the parametric surface $S : \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ with (u, v) in the region R is

$$\iint_S f dS = \iint_R f(u, v)|\mathbf{t}_u \times \mathbf{t}_v|dA$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v}$.

Definition. The **flux** of the vector field $\mathbf{F}(x, y, z)$ across the parametric surface $S : \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v$ with (u, v) in the region R is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA$$

Theorem. The **surface integral** of the scalar-valued function $f(x, y, z)$ over the surface $z = z(x, y)$ for (x, y) in the region R is

$$\iint_S f\ dS = \iint_R f(x, y, z(x, y))\sqrt{z_x^2 + z_y^2 + 1}\ dA$$

Theorem. The **flux** of the vector field $\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$ across the surface $z = z(x, y)$ for (x, y) in the region R is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iint_R (-f z_x - g z_y + h)\ dA$$

1. Let $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ and let S_1 be the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ and S_2 be the part of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$.

a) Plot the vector field and the surfaces in CalcPlot3D and make a guess about whether the flux of F across S_1 or across S_2 is greater.

b) Calculate the flux of $\mathbf{F}$ across S_1 and the flux of $\mathbf{F}$ across S_2.

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Theorem (Stokes' theorem)**.** Let S be an oriented surface with boundary C (the orientation of which must be consistent with the orientation of S.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl}\mathbf{F}) \cdot \mathbf{n}\, dS$$

2. Use Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $z + y = 2$ with the cylinder $x^2 + y^2 = 1$ (with a counterclockwise orientation when seen from above).