SUMMARY OF LAPLACE TRANSFORMS

1. Methods of solution

Method. To solve an IVP:
1) Take the Laplace transform of the differential equation;
2) Use the initial conditions;
3) Solve for $Y = L(y)$ (do as little algebra as possible);
4) Take the inverse Laplace transform to find the solution.

Method. To solve an IVP of the form $ay'' + by' + cy = f(t) + \alpha\delta(t - t_0)$, $y(0) = k_0$, $y'(0) = k_1$:
1) Find a solution $\hat{y}(t)$ to the IVP $ay'' + by' + cy = f(t)$, $y(0) = k_0$, $y'(0) = k_1$ (use any method);
2) Find the impulse response $w(t)$ of $ay'' + by' + cy = \delta(t - t_0)$ (definition below);
3) The solution to the original IVP is $\boxed{y(t) = \hat{y}(t) + \alpha u(t - t_0)w(t - t_0)}$.

Theorem. Let $t_0 \geq 0$. The solution to the IVP $ay'' + by' + cy = \delta(t - t_0)$, $y(0) = 0$, $y'(0) = 0$, is $\boxed{y = u(t - t_0)w(t - t_0)}$ where $\boxed{w(t) = L^{-1}\left(\dfrac{1}{as^2 + bs + c}\right)}$ ($w(t)$ is the **impulse response** of the system).

2. Theorems for Laplace transforms

Theorem. Suppose f and f' are continuous on $[0, \infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0, \infty)$. Then f, f', and f'' have Laplace trannsforms for $s > s_0$:

$$\boxed{L(f') = sL(f) - f(0)} \text{ and } \boxed{L(f'') = s^2L(f) - sf(0) - f'(0)}$$

Theorem (Linearity)**.** Let $c_1, c_2, \ldots, c_n$ be constants and let $f_1, f_2, \ldots, f_n$ and $F_1, F_2, \ldots, F_n$ be functions. Then

$$L(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = c_1L(f_1) + c_2L(f_2) + \cdots + c_nL(f_n)$$

and

$$L^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1L^{-1}(F_1) + c_2L^{-1}(F_2) + \cdots + c_nL^{-1}(F_n)$$

Theorem (Superposition)**.** If y_1 is the solution of the IVP

$$ay'' + by' + cy = f_1(t),\ y(0) = k_1,\ y'(0) = k_2$$

and y_2 is the solution of the IVP

$$ay'' + by' + cy = f_2(t),\ y(0) = l_1,\ y'(0) = l_2,$$

then $y_1 + y_2$ is the solution to the IVP

$$ay'' + by' + cy = f_1(t) + f_2(t),\ y(0) = k_1 + l_1,\ y'(0) = k_2 + l_2.$$

Theorem (First Shifting Theorem)**.** If $L(f(t)) = F(s)$ for $s > s_0$, then $\boxed{L\left(e^{at}f(t)\right) = F(s - a)}$ for $s > s_0 + a$.

Theorem (Second shifting theorem)**.** $\boxed{L^{-1}\left(e^{-st_0}L(g(t))\right) = u(t - t_0)g(t - t_0)}$

Theorem. If $t_0 \geq 0$ and $L(g(t + t_0))$ exists for $s > s_0$, then $\boxed{L(u(t - t_0)g(t)) = e^{-st_0}L(g(t + t_0))}$ for $s > s_0$.

Theorem (Convolution Theorem)**.** $\boxed{L(f * g) = L(f)L(g)}$

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3. Special functions and operations

Definition. The **unit step function** (or **Heaviside function**) is $u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$

Method. The piecewise function $f(t) = \begin{cases} g(t) & \text{if } t < t_1 \\ h(t) & \text{if } t \geq t_1 \end{cases}$ can be expressed as $f(t) = g(t) + u(t - t_1)\,[h(t) - g(t)]$

Definition. Let f and g be functions such that if $t < 0$, then $f(t) = 0$ and $g(t) = 0$. The **convolution** of f and g is the function $f * g$ defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

Definition. The **Dirac delta** (or **unit impulse**) function is defined to be the "function" $\delta(t)$ such that

(1) $\delta(t) = 0$ if $t \neq 0$

(2) $\displaystyle\int_{\infty}^{\infty} \delta(t)dt = 1$

Note that there is no real number y such that $\delta(0) = y$.

Definition. The **Laplace transform** of $f(t)$ is $\int_0^\infty e^{-st} f(t)dt$. Note, however, that this is rarely used. Instead, refer to the a table of Laplace transforms (see section 8.8 of the textbook).