Instructions: The exam consists of K total questions, N in section 1 and $K-N$ in section 2. Solve $N-1$ of the N section 1 problems and **all** of the section 1 problems. Write your solutions to the section 1 problems on the provided paper, clearly labeling each solution. Write your solutions to the section 2 problems on the problem page. Calculators, phones, and all other devices are forbidden. Answers may be left unsimplified.

Thoerem. *The general solution to the homogeneous linear differential equation $y' + p(x)y = 0$ is*

$$\boxed{y = ce^{-P(x)}}$$

where $P'(x) = p(x)$.

Thoerem. *The general solution to the linear differential equation $y' + p(x)y = f(x)$ is $\boxed{y = uy_1}$ where*

a) y_1 is any particular solution to the complementary equation $y' + p(x)y = 0$ and

b) $u = \int \frac{f(x)}{y_1(x)} dx$ (add a constant here).

Conversion to the Ponicaré phase plane: $y' = v$ and $y'' = v\frac{dv}{dy}$.

Thoerem. *If the characteristic polynomial of $ay'' + by' + cy' = 0$ has...*

a) ...distinct real roots r_1 and r_2, then a general solution is $y = c_1e^{r_1x} + c_2e^{r_2x}$

b) ...a single (repeated) real root r, then a general solution is $y = e^{rx}(c_1 + c_2x)$

c) ...complex conjugate roots $\lambda \pm i\omega$ (where $\omega > 0$), then a general solution is $y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$

Thoerem. *If y_p is any particular solution to the differential equation $y'' + p(x)y' + q(x)y = f(x)$ and $\{y_1, y_2\}$ is a fundamental set of solutions to the complementary equation, then a general solution for differential equation is*

$$y = y_p + c_1y_1 + c_2y_2$$

Thoerem (Superposition)**.** *If y_{p_1} is a particular solution of $y'' + p(x)y' + q(x)y = f_1(x)$ and y_{p_2} is a particular solution of $y'' + p(x)y' + q(x)y = f_2(x)$, then a particular solution of*

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

is

$$y_p = y_{p_1} + y_{p_2}$$

Method. To find a particular solution for $ay'' + by' + cy = e^{\alpha x}G(x)$ (where G is a polynomial), solve for the coefficients of $Q(x)$ in the following (where Q is a polynomial with the same degree as G):

a) $y_p = e^{\alpha x}Q(x)$ if $e^{\alpha x}$ is not a solution to the complementary equation;

b) $y_p = xe^{\alpha x}Q(x)$ if $xe^{\alpha x}$ is not a solution to the complementary equation, but $e^{\alpha x}$ is a solution to the complementary equation;

c) $y_p = x^2e^{\alpha x}Q(x)$ if $xe^{\alpha x}$ and $e^{\alpha x}$ are both solutions to the complementary equation.

Method. To find a particular solution for $ay'' + by' + cy = P(x)\cos \omega x + Q(x)\sin \omega x$ (where P and Q are polynomials), solve for the coefficients of $A(x)$ and $B(x)$ in the following (where A and B are polynomials with degree equal to the larger of the degrees of P and Q):

a) $y_p = A(x)\cos \omega x + B(x)\sin \omega x$ if $\cos \omega x$ and $\sin \omega x$ are not solutions to the complementary equation;

b) $y_p = x\left[A(x)\cos\omega x + B(x)\sin\omega x\right]$ if $\cos\omega x$ and $\sin\omega x$ are solutions to the complementary equation;

Definition. The **Laplace transform** of f is $\boxed{L(f) = F(s) = \int_0^\infty f(t)e^{-st}dt}$

Thoerem (First Shifting Theorem)**.** *If* $L(f) = F(s)$*, then* $\boxed{L\left(e^{at}f(t)\right) = F(s-a)}$

Thoerem. $\boxed{L(f') = sL(f) - f(0)}$ *and* $\boxed{L(f'') = s^2L(f) - sf(0) - f'(0)}$

Thoerem. $f(t) = \begin{cases} f_0(t), & 0 \le t < t_0 \\ f_1(t), & t \ge t_0 \end{cases} = f_0(t) + u(t-t_0)\left[f_1(t) - f_0(t)\right]$

Thoerem (Second Shifting Theorem)**.**

1. $\boxed{L\left(u(t-t_0)g(t)\right) = e^{-t_0 s}L\left(g(t+t_0)\right)}$
2. $\boxed{L\left(u(t-t_0)g(t-t_0)\right) = e^{-t_0 s}L\left(g\right)}$

Definition. The convolution of functions f and g is the function $f * g$ defined by $\boxed{(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau}$

Thoerem (Convolution Theorem)**.** *If* $L(f) = F$ *and* $L(g) = G$*, then* $\boxed{L(f*g) = FG}$

Definition. The solution to the initial value problem $ay'' + by' + cy = \delta(t-t_0),\ y(0) = 0,\ y'(0) = 0$ is $\boxed{y = u(t-t_0)w(t-t_0)}$ where $w(t) = L^{-1}\left(\dfrac{1}{as^2 + bs + c}\right)$.

Thoerem (Superposition)**.** *If* y_1 *is the solution of the IVP*

$$ay'' + by' + cy = f_1(t),\ y(0) = k_1,\ y'(0) = k_2$$

and y_2 *is the solution of the IVP*

$$ay'' + by' + cy = f_2(t),\ y(0) = l_1,\ y'(0) = l_2,$$

then $y_1 + y_2$ *is the solution to the IVP*

$$ay'' + by' + cy = f_1(t) + f_2(t),\ y(0) = k_1 + l_1,\ y'(0) = k_2 + l_2.$$

Laplace transforms

$f(t)$	1	t^n	e^{at}	$t^n e^{at}$	$\sin \omega t$	$\cos \omega t$	$\sinh bt$	$\cosh bt$	$\delta(t - t_0)$
$F(S)$	$\frac{1}{s}$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{n!}{(s-a)^{n+1}}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$	$\frac{b}{s^2-b^2}$	$\frac{s}{s^2-b^2}$	$e^{-t_0 s}$

Thoerem. *Let A be an $n \times n$ matrix with real entries.*

i) If A has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with associated **linearly independent** *eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then the functions*

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t} \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_2 t} \\ &\vdots \\ \mathbf{y}_n &= \mathbf{x}_n e^{\lambda_n t} \end{aligned}$$

form a fundamental set of solutions of $A\mathbf{y} = \mathbf{y}'$.

ii) If A has an eigenvalue λ with multiplicity of 2 or more and with an associated eigenspace of dimension 1, then there are infinitely many vectors $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{x}$. If $\mathbf{u}$ is any such vector, then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x} e^{\lambda t} \\ \mathbf{y}_2 &= \mathbf{x} t e^{\lambda t} + \mathbf{u} e^{\lambda t} \end{aligned}$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.

a) If λ has multiplicity 3 or more and has an associated eigenspace of dimension 1, then in addition to the preceding there are infinitely many vectors $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{u}$. If $\mathbf{v}$ is any such vector, then an additional linearly independent solution is

$$\mathbf{y}_3 = \mathbf{v} e^{\lambda t} + \mathbf{u} t e^{\lambda t} + \mathbf{x} \left(\frac{t^2}{2} \right) e^{\lambda t}$$

b) If λ has multiplicity 3 or more and has an associated eigenspace of dimension 2, then see page 550 of the textbook for a solution.

iii) If A has a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, then both $\mathbf{u}$ and $\mathbf{v}$ are nonzero and

$$\begin{aligned} \mathbf{y}_1 &= e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \\ \mathbf{y}_2 &= e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned}$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.

Method. To solve the IVP $A\mathbf{y} = \mathbf{y}'$, $\mathbf{y}(0) = \mathbf{b}$:

1. Find the eigenvalues of A.
2. Find an eigenvector for each eigenvalue.
3. Use the theorem above to find a fundamental set of solutions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$.
4. Your general solution is $\mathbf{y}(t) = c_1 \mathbf{y}_1(t) + c_2 \mathbf{y}_2(t) + \cdots + c_n \mathbf{y}_n(t)$.

5. Use the initial condition to solve the system of equations for $c_1, c_2, \ldots, c_n$:
   $\mathbf{b} = \mathbf{y}(0) = c_1\mathbf{y}_1(0) + c_2\mathbf{y}_2(0) + \cdots + c_n\mathbf{y}_n(0)$.

Method. The eigenvalues of an $n \times n$ matrix A are the solutions to $\det(A - \lambda I) = 0$. Find the eigenvector(s) corresponding to eigenvalue λ by solving $(A - \lambda I)\mathbf{x} = 0$. In 2-dimensional systems, the solution to $(A - \lambda I)\mathbf{x} = 0$ is usually a line $ax_1 + bx_2 = 0$; to find an eigenvalue, just choose a value for x_1 or x_2 and solve for the other.

Section 1 problems

Instructions: Solve $N - 1$ of the section 1 problems and place an **X** in the box below to indicate which you are skipping. Write your solutions to the section 1 problems on the provided paper, clearly labeling each solution. Solutions to section 1 problems should include a clear method or argument and should use English words and sentences when appropriate. Clear and comprehensible solutions will generally earn more points than those that are hard to understand; a correct solution without supporting work may receive little or no credit.

1	2	3	4	5	6	...	N	Section 1 Total

Section 2 problems

Instructions: Solve all $K - N$ of the section 2 problems. Write your solutions on these pages.