initial values $v_0 = 0, -5, -10, \dots, -30$. Present your results in graphical form similar to Figure 4.3.3.

In Exercises 17-20, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

19. **(a)** Show that Eqn. (4.3.23) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

(b) Show that if $v_0 = \rho v_e$ with $0 \le \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1-\rho^2}.$$

(c) By requiring that $v(y_m) = 0$, use Eqn. (4.3.22) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1-\rho^2)(y_m - y)}{y+R} \right]^{1/2},$$

where y_m and ρ are as defined in **(b)** and $y \ge h$.

(d) Deduce from **(c)** that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

HINT: *Use a proof by contradiction. Assume that there's a number y_m such that $y(t) \le y_m$ for all $t > 0$. Deduce from this that there's positive number α such that $y''(t) \le -\alpha$ for all $t \ge 0$. Show that this contradicts the assumption that $v(t) > 0$ for all $t > 0$.*

4.4 AUTONOMOUS SECOND ORDER EQUATIONS

A second order differential equation that can be written as

$$y'' = F(y, y') \tag{4.4.1}$$

where F is independent of t, is said to be *autonomous*. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and y. If we let $v = y'$, (4.4.1) becomes

$$v' = F(y, v). \tag{4.4.2}$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}, \tag{4.4.3}$$

(4.4.2) can be rewritten as

$$v\frac{dv}{dy} = F(y,v). \tag{4.4.4}$$

The integral curves of (4.4.4) can be plotted in the (y,v) plane, which is called the *Poincaré phase plane* of (4.4.1). If y is a solution of (4.4.1) then $y = y(t), v = y'(t)$ is a parametric equation for an integral curve of (4.4.4). We'll call these integral curves *trajectories* of (4.4.1), and we'll call (4.4.4) the *phase plane equivalent* of (4.4.1).

In this section we'll consider autonomous equations that can be written as

$$y'' + q(y,y')y' + p(y) = 0. \tag{4.4.5}$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose y is the displacement of a moving object with mass m. It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called *damping* forces.) We write this force as $-mq(y,y')y'$, where $q(y,y')$ is usually a positive function and we've put the factor y' outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to (4.4.5).

The phase plane equivalent of (4.4.5) is

$$v\frac{dv}{dy} + q(y,v)v + p(y) = 0. \tag{4.4.6}$$

Some statements that we'll be making about the properties of (4.4.5) and (4.4.6) are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we'll just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all y and $q = q(y,v)$ is continuously differentiable for all (y,v). We begin with the following statements:

- **Statement 1.** If y_0 and v_0 are arbitrary real numbers then (4.4.5) has a unique solution on $(-\infty,\infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.

- **Statement 2.)** If $y = y(t)$ is a solution of (4.4.5) and τ is any constant then $y_1 = y(t-\tau)$ is also a solution of (4.4.5), and y and y_1 have the same trajectory.

- **Statement 3.** If two solutions y and y_1 of (4.4.5) have the same trajectory then $y_1(t) = y(t-\tau)$ for some constant τ.

- **Statement 4.** Distinct trajectories of (4.4.5) can't intersect; that is, if two trajectories of (4.4.5) intersect, they are identical.

- **Statement 5.** If the trajectory of a solution of (4.4.5) is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time T, and the solution is periodic with period T; that is, $y(t+T) = y(t)$ for all t in $(-\infty,\infty)$.

If $\overline{y}$ is a constant such that $p(\overline{y}) = 0$ then $y \equiv \overline{y}$ is a constant solution of (4.4.5). We say that $\overline{y}$ is an *equilibrium* of (4.4.5) and $(\overline{y},0)$ is a *critical point* of the phase plane equivalent equation (4.4.6). We say that the equilibrium and the critical point are *stable* if, for any given $\epsilon > 0$ *no matter how small*, there's a $\delta > 0$, *sufficiently small*, such that if

$$\sqrt{(y_0-\overline{y})^2 + v_0^2} < \delta$$

then the solution of the initial value problem

$$y'' + q(y, y')y' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

satisfies the inequality

$$\sqrt{(y(t) - \overline{y})^2 + (v(t))^2} < \epsilon$$

for all $t > 0$. Figure 4.4.1 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if (y_0, v_0) is in the smaller shaded circle (with radius δ), then $(y(t), v(t))$ must be in in the larger circle (with radius ϵ) for all $t > 0$.

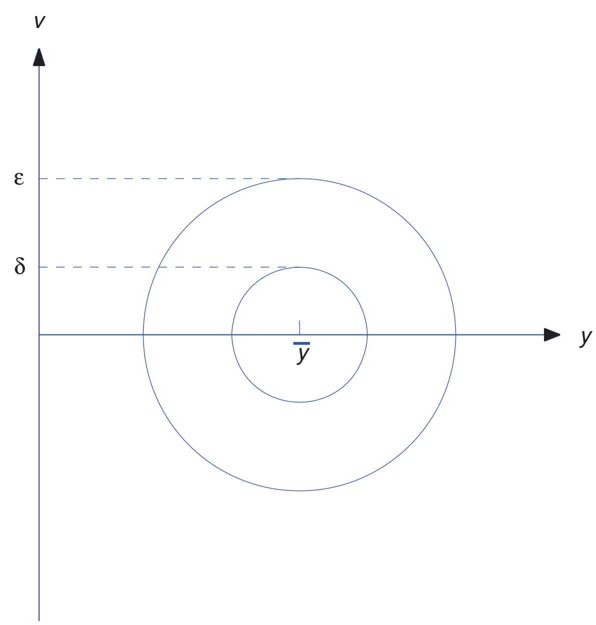

Figure 4.4.1 Stability: if (y_0, v_0) is in the smaller circle then $(y(t), v(t))$ is in the larger circle for all $t > 0$

If an equilibrium and the associated critical point are not stable, we say they are *unstable*. To see if you really understand what *stable* means, try to give a direct definition of *unstable* (Exercise 22). We'll illustrate both definitions in the following examples.

The Undamped Case

We'll begin with the case where $q \equiv 0$, so (4.4.5) reduces to

$$y'' + p(y) = 0. \tag{4.4.7}$$

We say that this equation - as well as any physical situation that it may model - is *undamped*. The phase plane equivalent of (4.4.7) is the separable equation

$$v\frac{dv}{dy} + p(y) = 0.$$

Integrating this yields

$$\frac{v^2}{2} + P(y) = c, \tag{4.4.8}$$

where c is a constant of integration and $P(y) = \int p(y)\,dy$ is an antiderivative of p.

If (4.4.7) is the equation of motion of an object of mass m, then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, (4.4.8) says that the total energy of the object remains constant, or is *conserved*. In particular, if a trajectory passes through a given point (y_0, v_0) then

$$c = \frac{v_0^2}{2} + P(y_0).$$

Example 4.4.1 [*The Undamped Spring - Mass System*] Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2).

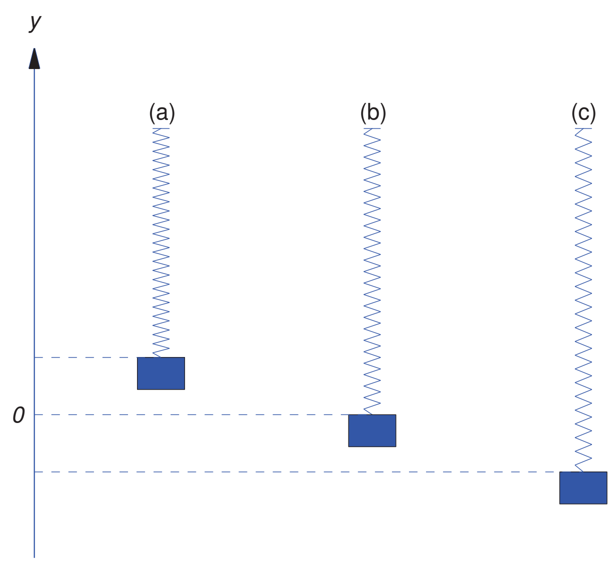

Figure 4.4.2 **(a)** $y > 0$ **(b)** $y = 0$ **(c)** $y < 0$

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1 it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \tag{4.4.9}$$

which can be written in the form (4.4.7) with $p(y) = ky/m$. This equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider the phase plane equivalent of (4.4.9).

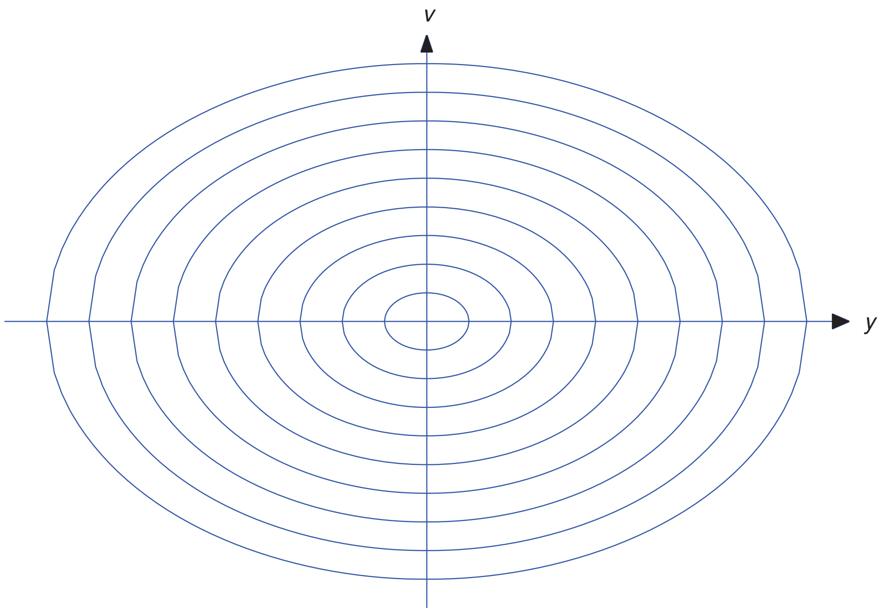

Figure 4.4.3 Trajectories of $my'' + ky = 0$

From (4.4.3), we can rewrite (4.4.9) as the separable equation

$$mv\frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \tag{4.4.10}$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3).

We can identify ρ by setting $t = 0$ in (4.4.10); thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in (4.4.10); thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with } R = \sqrt{\frac{\rho}{k}}. \tag{4.4.11}$$

Equation (4.4.9) has exactly one equilibrium, $\overline{y} = 0$, and it's stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t. (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I, (4.4.10) implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

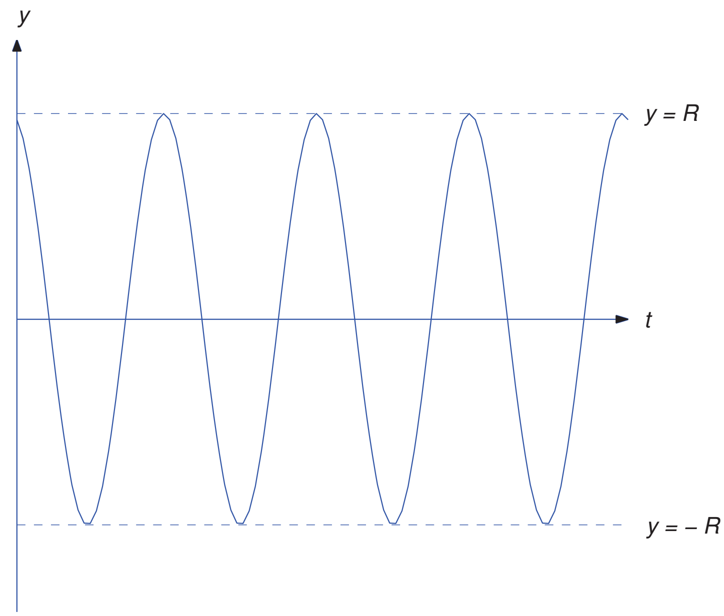

Figure 4.4.4 $y = R\sin(\omega_0 t + \phi)$

on I. This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}}\frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4.4.12}$$

Since

$$\int \frac{\sqrt{k}\,dy}{\sqrt{\rho - ky^2}} = \sin^{-1}\left(\sqrt{\frac{k}{\rho}}y\right) + c = \sin^{-1}\left(\frac{y}{R}\right) + c$$

(see (4.4.11)), (4.4.12) implies that that there's a constant ϕ such that

$$\sin^{-1}\left(\frac{y}{R}\right) = \omega_0 t + \phi$$

or

$$y = R\sin(\omega_0 t + \phi)$$

for all t in I. Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies (4.4.9) for all values of t. Thus, the displacement varies periodically between $-R$ and R, with period $T = 2\pi/\omega_0$ (Figure 4.4.4). (If you've taken a course in elementary mechanics you may recognize this as *simple harmonic motion*.)

Example 4.4.2 [*The Undamped Pendulum*] Now we consider the motion of a pendulum with mass m, attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 4.4.5). We assume that there's no air resistance.

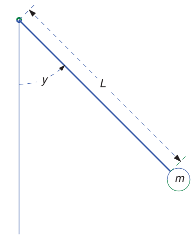

Figure 4.4.5 The undamped pendulum

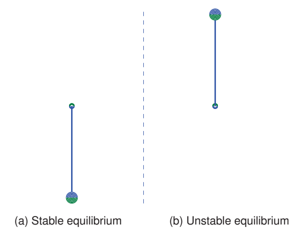

Figure 4.4.6 **(a)** Stable equilibrium **(b)** Unstable equilibrium

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5. Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5,

$$mLy'' = -mg\sin y,$$

or

$$y'' = -\frac{g}{L}\sin y. \tag{4.4.13}$$

Since $\sin n\pi = 0$ if n is any integer, (4.4.13) has infinitely many equilibria $\overline{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure 4.4.6 **(a)**) and gravity opposes any deviation from the equilibrium. However, if n is odd, the mass is directly above the axle (Figure 4.4.6 **(b)**) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\overline{y}_{2m} = 2m\pi$ is stable and $\overline{y}_{2m+1} = (2m+1)\pi$ is unstable.

The phase plane equivalent of (4.4.13) is

$$v\frac{dv}{dy} = -\frac{g}{L}\sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L}\cos y + c. \tag{4.4.14}$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so (4.4.14) becomes

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{g}{L}(1-\cos y) = \frac{v_0^2}{2} - \frac{2g}{L}\sin^2\frac{y}{2},$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2 \sin^2\frac{y}{2}, \tag{4.4.15}$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by (4.4.15) are the trajectories of (4.4.13). They are periodic with period 2π in y, which isn't surprising, since if $y = y(t)$ is a solution of (4.4.13) then so is $y_n = y(t) + 2n\pi$ for any integer n. Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$. From (4.4.15), you can see that if $|v_0| > v_c$ then v is nonzero for all t, which means that the object whirls in the same direction forever, as in Figure 4.4.8. The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7. You can also see from (4.4.15) that if $0 < |v_0| < v_c$,then $v = 0$ when $y = \pm y_{\max}$, where

$$y_{\max} = 2\sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9. The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 4.4.7. It can be shown (see Exercise 21 for a partial proof) that the period of the oscillation is

$$T = 8\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2\sin^2\theta}}. \tag{4.4.16}$$

Although this integral can't be evaluated in terms of familiar elementary functions, you can see that it's finite if $|v_0| < v_c$.

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\overline{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = \pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = -\pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = \pi.$$

Consistent with this, the integral (4.4.16) diverges to ∞ if $v_0 = \pm v_c$. (Exercise 21) .

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a *separatrix*.

In general, if (4.4.7) has both stable and unstable equilibria then the separatrices are the curves given by (4.4.8) that pass through unstable critical points. Thus, if $(\overline{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\overline{y}) \tag{4.4.17}$$

defines a separatrix passing through $(\overline{y}, 0)$.

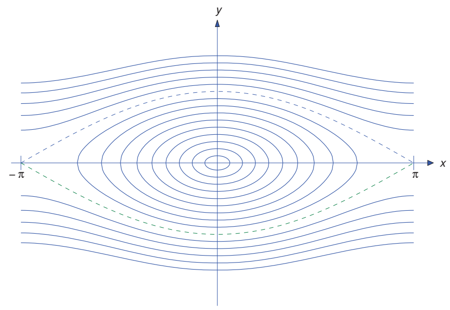

Figure 4.4.7 Trajectories of the undamped pendulum

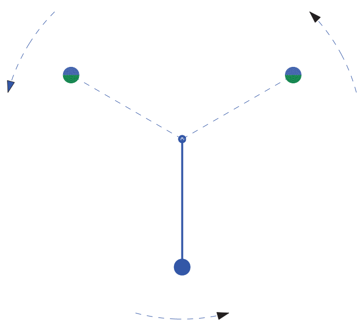

Figure 4.4.8 The whirling undamped pendulum

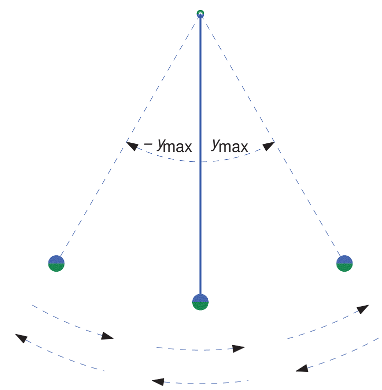

Figure 4.4.9 The oscillating undamped pendulum

Stability and Instability Conditions for $y'' + p(y) = 0$

It can be shown (Exercise 23) that an equilibrium $\overline{y}$ of an undamped equation

$$y'' + p(y) = 0 \tag{4.4.18}$$

is stable if there's an open interval (a, b) containing $\overline{y}$ such that

$$p(y) < 0 \text{ if } a < y < \overline{y} \text{ and } p(y) > 0 \text{ if } \overline{y} < y < b. \tag{4.4.19}$$

If we regard $p(y)$ as a force acting on a unit mass, (4.4.19) means that the force resists all sufficiently small displacements from $\overline{y}$.

We've already seen examples illustrating this principle. The equation (4.4.9) for the undamped spring-mass system is of the form (4.4.18) with $p(y) = ky/m$, which has only the stable equilibrium $\overline{y} = 0$. In this case (4.4.19) holds with $a = -\infty$ and $b = \infty$. The equation (4.4.13) for the undamped pendulum is of the form (4.4.18) with $p(y) = (g/L)\sin y$. We've seen that $\overline{y} = 2m\pi$ is a stable equilibrium if m is an integer. In this case

$$p(y) = \sin y < 0 \text{ if } (2m-1)\pi < y < 2m\pi$$

and

$$p(y) > 0 \text{ if } 2m\pi < y < (2m+1)\pi.$$

It can also be shown (Exercise 24) that $\overline{y}$ is unstable if there's a $b > \overline{y}$ such that

$$p(y) < 0 \text{ if } \overline{y} < y < b \tag{4.4.20}$$

or an $a < \overline{y}$ such that

$$p(y) > 0 \text{ if } a < y < \overline{y}. \tag{4.4.21}$$

If we regard $p(y)$ as a force acting on a unit mass, (4.4.20) means that the force tends to increase all sufficiently small positive displacements from $\overline{y}$, while (4.4.21) means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\overline{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\overline{y} = (2m+1)\pi$ is an unstable equilibrium if m is an integer. In this case

$$\sin y < 0 \text{ if } (2m+1)\pi < y < (2m+2)\pi,$$

so (4.4.20) holds with $b = (2m+2)\pi$, and

$$\sin y > 0 \text{ if } 2m\pi < y < (2m+1)\pi,$$

so (4.4.21) holds with $a = 2m\pi$.

Example 4.4.3 The equation

$$y'' + y(y-1) = 0 \tag{4.4.22}$$

is of the form (4.4.18) with $p(y) = y(y-1)$. Therefore $\overline{y} = 0$ and $\overline{y} = 1$ are the equilibria of (4.4.22). Since

$$\begin{aligned} y(y-1) &> 0 \quad \text{if } y < 0 \text{ or } y > 1, \\ &< 0 \quad \text{if } 0 < y < 1, \end{aligned}$$

$\overline{y} = 0$ is unstable and $\overline{y} = 1$ is stable.

The phase plane equivalent of (4.4.22) is the separable equation

$$v\frac{dv}{dy} + y(y-1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

$$v^2 + \frac{1}{3}y^2(2y-3) = c \tag{4.4.23}$$

after renaming the constant of integration. These are the trajectories of (4.4.22). If y is any solution of (4.4.22), the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in (4.4.23), separates the y-v plane into regions that contain different kinds of trajectories; again, we call this curve a *separatrix*. Trajectories in the region bounded by the closed loop **(b)** are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, then

$$\begin{aligned} \lim_{t\to\infty} y(t) &= -\infty, & \lim_{t\to-\infty} y(t) &= -\infty, \\ \lim_{t\to\infty} v(t) &= -\infty, & \lim_{t\to-\infty} v(t) &= \infty. \end{aligned}$$

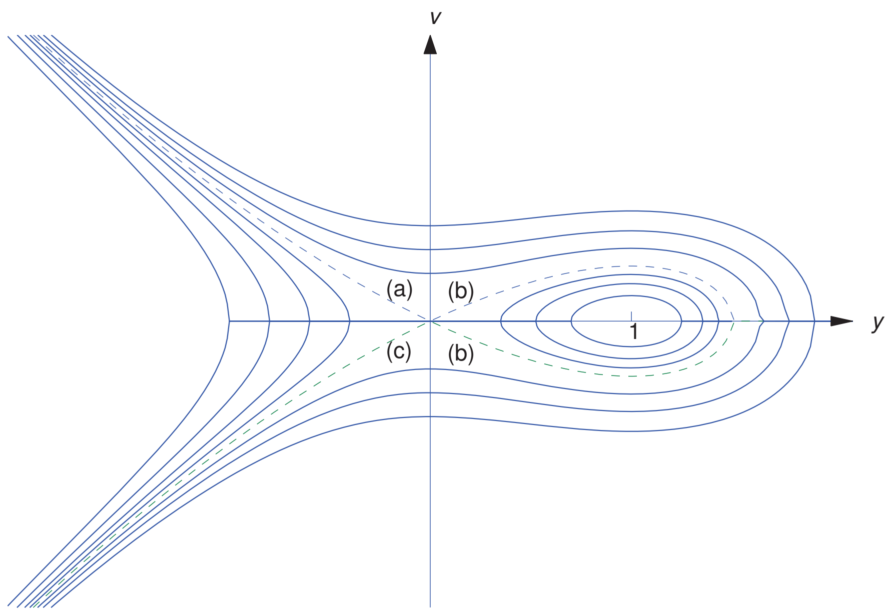

Figure 4.4.10 Trajectories of $y'' + y(y-1) = 0$

The separatrix contains four trajectories of (4.4.22). One is the point $(0,0)$, the trajectory of the equilibrium $\overline{y} = 0$. Since distinct trajectories can't intersect, the segments of the separatrix marked **(a)**, **(b)**, and **(c)** – which don't include $(0,0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{array}{rclrcll}
\lim_{t\to\infty} y(t) & = & 0, & \lim_{t\to-\infty} y(t) & = & -\infty, & \\
\lim_{t\to\infty} v(t) & = & 0, & \lim_{t\to-\infty} v(t) & = & \infty & \text{(on } \mathbf{(a)}\text{)} \\
\lim_{t\to\infty} y(t) & = & 0, & \lim_{t\to-\infty} y(t) & = & 0, & \\
\lim_{t\to\infty} v(t) & = & 0, & \lim_{t\to-\infty} v(t) & = & 0 & \text{(on } \mathbf{(b)}\text{)} \\
\lim_{t\to\infty} y(t) & = & -\infty, & \lim_{t\to-\infty} y(t) & = & 0, & \\
\lim_{t\to\infty} v(t) & = & -\infty, & \lim_{t\to-\infty} v(t) & = & 0 & \text{(on } \mathbf{(c)}\text{).}
\end{array}$$

The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y, y')y' + p(y) = 0 \tag{4.4.24}$$

is

$$v\frac{dv}{dy} + q(y, v)v + p(y) = 0.$$

This equation isn't separable, so we can't solve it for v in terms of y, as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy} = -q(y, v) - \frac{p(y)}{v} \tag{4.4.25}$$

and considering the direction fields for this equation. In the following examples we'll also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in (4.4.25) is undefined on the y axis of the Poincaré phase plane. Therefore we're forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned} y' &= v \\ v' &= -q(y, v)v - p(y), \end{aligned}$$

which is equivalent to (4.4.24) in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we'll confine ourselves to the case where q is constant, so (4.4.24) and (4.4.25) reduce to

$$y'' + cy' + p(y) = 0 \tag{4.4.26}$$

and

$$\frac{dv}{dy} = -c - \frac{p(y)}{v}.$$

(We'll consider more general equations in the exercises.) The constant c is called the *damping constant*. In situations where (4.4.26) is the equation of motion of an object, c is positive; however, there are situations where c may be negative.

The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we'll assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my'' + cy' + ky = 0, \tag{4.4.27}$$

where $c > 0$ is the *damping constant*. Again, this equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider its phase plane equivalent, which can be written in the form (4.4.25) as

$$\frac{dv}{dy} = -\frac{c}{m} - \frac{ky}{mv}. \tag{4.4.28}$$

(A minor note: the c in (4.4.26) actually corresponds to c/m in this equation.) Figure 4.4.11 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing y in the upper half plane ($v > 0$) and in the direction of decreasing y in the lower half plane ($v < 0$), you can infer that all trajectories approach the origin in clockwise fashion. To confirm this, Figure 4.4.12 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad \text{(a positive constant)};$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 4.4.14.

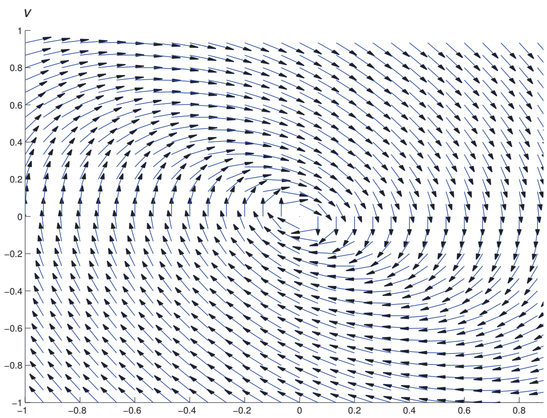

Figure 4.4.11 A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$

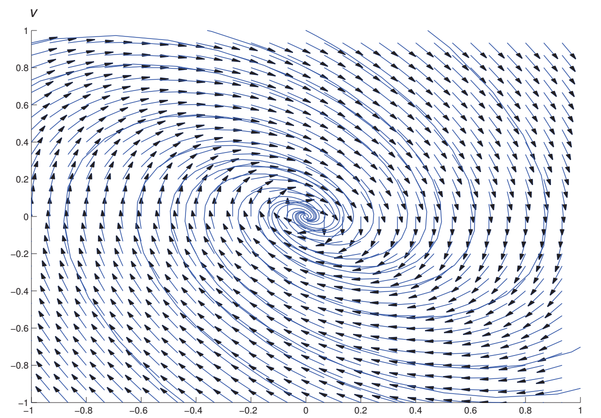

Figure 4.4.12 Figure 4.4.11 with some trajectories added

Solutions corresponding to the trajectories in Figure 4.4.12 cross the y-axis infinitely many times. The corresponding solutions are said to be *oscillatory* (Figure 4.4.13) It is shown in Section 6.2 that there's a number c_1 such that if $0 \le c < c_1$ then all solutions of (4.4.27) are oscillatory, while if $c \ge c_1$, no solutions of (4.4.27) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 4.4.14 shows a direction field and some integral curves for (4.4.28) in this case.

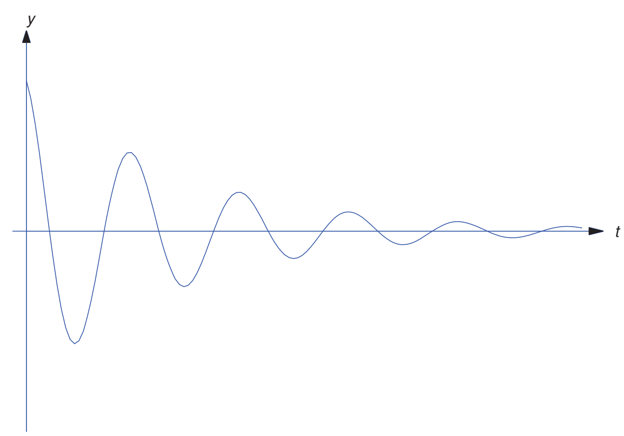

Figure 4.4.13 An oscillatory solution of $my'' + cy' + ky = 0$

Example 4.4.4 (The Damped Pendulum) Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg \sin y, \tag{4.4.29}$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in (4.4.26) actually corresponds to

c/mL in this equation.) To plot a direction field for (4.4.29) we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv}\sin y.$$

Figure 4.4.15 shows trajectories of four solutions of (4.4.29), all satisfying $y(0) = 0$. For each $m = 0, 1, 2, 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\overline{y} = 2m\pi$.

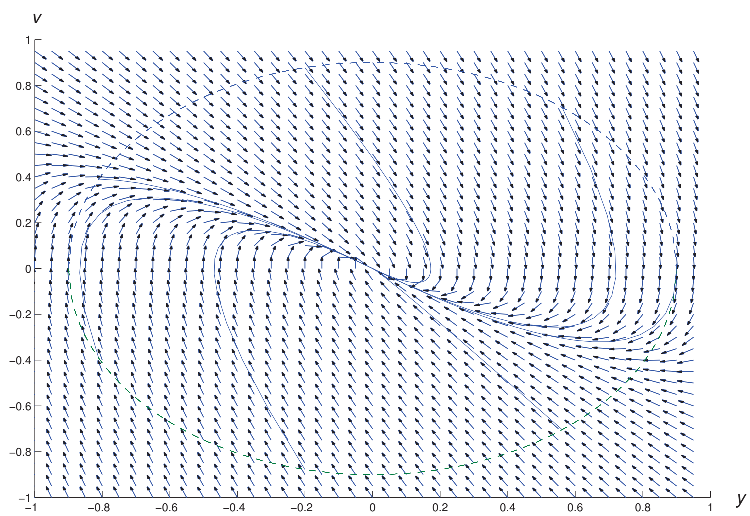

Figure 4.4.14 A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$

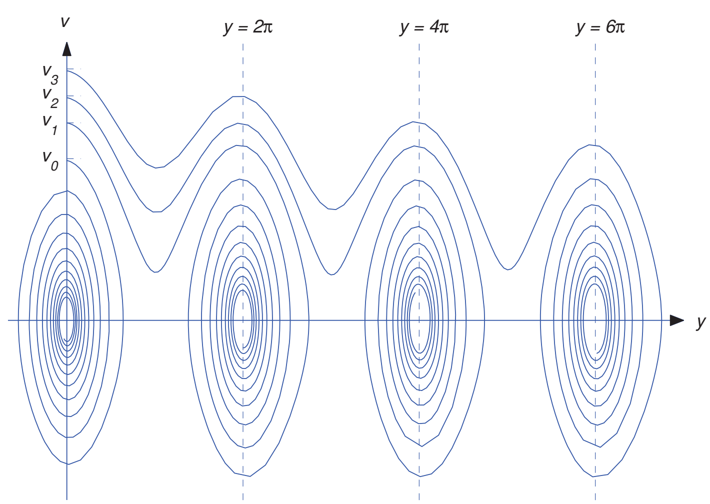

Figure 4.4.15 Four trajectories of the damped pendulum

4.4 Exercises

In Exercises 1–4 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories. HINT: *Use Eqn. (4.4.8) to obtain the equations of the trajectories.*

1. C/G $y'' + y^3 = 0$

2. C/G $y'' + y^2 = 0$

3. C/G $y'' + y|y| = 0$

4. C/G $y'' + ye^{-y} = 0$

In Exercises 5–8 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them. HINT: *Use Eqn. (4.4.17) to determine the separatrices.*

5. C/G $y'' - y^3 + 4y = 0$

6. C/G $y'' + y^3 - 4y = 0$

7. C/G $y'' + y(y^2 - 1)(y^2 - 4) = 0$

8. C/G $y'' + y(y - 2)(y - 1)(y + 2) = 0$

In Exercises 9–12 plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter a. Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the critical value of a?

9. L $y'' + y^2 - a = 0$

10. L $y'' + y^3 - ay = 0$

11. L $y'' - y^3 + ay = 0$

12. L $y'' + y - ay^3 = 0$

In Exercises 13-18 plot trajectories of the given equation for $c = 0$ and small nonzero (positive and negative) values of c to observe the effects of damping.

13. L $y'' + cy' + y^3 = 0$

14. L $y'' + cy' - y = 0$

15. L $y'' + cy' + y^3 = 0$

16. L $y'' + cy' + y^2 = 0$

17. L $y'' + cy' + y|y| = 0$

18. L $y'' + y(y - 1) + cy = 0$

19. L The *van der Pol equation*

$$y'' - \mu(1 - y^2)y' + y = 0, \tag{A}$$

where μ is a positive constant and y is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term $-\mu(1 - y^2)y'$ works to reduce $|y|$ if $|y| < 1$ or to increase $|y|$ if $|y| > 1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a *limit cycle*. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure 4.4.16). Use your favorite differential equations software to verify this for $\mu = .5, 1.1.5, 2$. Use a grid with $-4 < y < 4$ and $-4 < v < 4$.

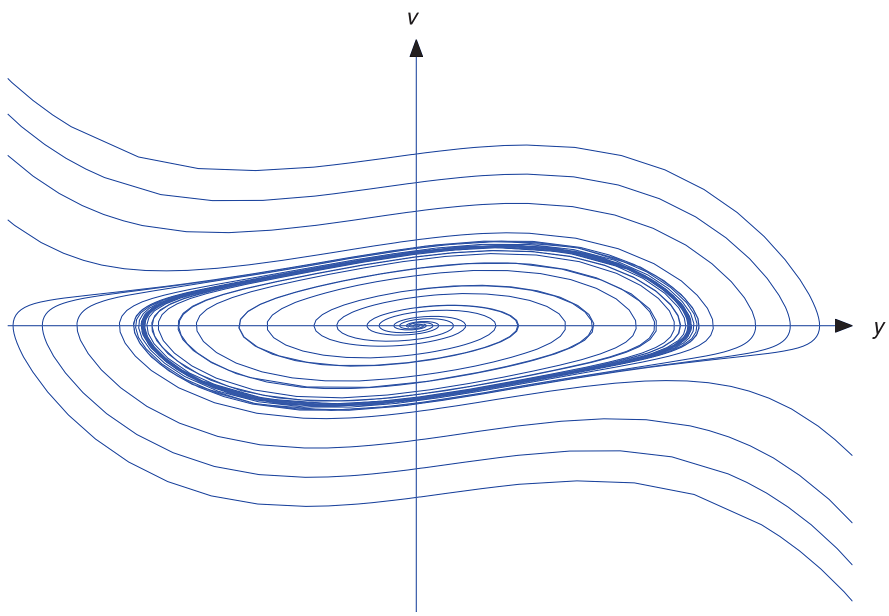

Figure 4.4.16 Trajectories of van der Pol's equation

20. L *Rayleigh's equation*,

$$y'' - \mu(1 - (y')^2/3)y' + y = 0$$

also has a limit cycle. Follow the directions of Exercise 19 for this equation.

21. In connection with Eqn (4.4.15), suppose $y(0) = 0$ and $y'(0) = v_0$, where $0 < v_0 < v_c$.

(a) Let T_1 be the time required for y to increase from zero to $y_{\max} = 2\sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt} = \sqrt{v_0^2 - v_c^2 \sin^2 y/2}, \quad 0 \le t < T_1. \tag{A}$$

(b) Separate variables in (A) and show that

$$T_1 = \int_0^{y_{\max}} \frac{du}{\sqrt{v_0^2 - v_c^2 \sin^2 u/2}} \tag{B}$$

(c) Substitute $\sin u/2 = (v_0/v_c)\sin\theta$ in (B) to obtain

$$T_1 = 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2\theta}}. \tag{C}$$