Theorem. The general solution to the homogeneous linear differential equation $y' + p(x)y = 0$ is

$$\boxed{y = ce^{-P(x)}}$$

where $P'(x) = p(x)$.

Note that applying the theorem requires that $p(x)$ be integrable; thus there are situations in which the theorem doesn't help.

1. Find the general solution to the differential equation $(1 + x^2)y' = 2xy$.

Rewrite as $y' - \left(\frac{2x}{1+x^2}\right)y = 0$. The equation is homog. linear with $p(x) = -\frac{2x}{1+x^2}$.

An antiderivative is $P(x) = -\ln(1+x^2)$ (no absolute values needed because $1+x^2 \geq 1$ for all x).

Thus the general solution is $\boxed{y = ce^{\ln(1+x^2)} = c(1+x^2)}$.

Check: $y' = 2cx$, so $(1+x^2)y' = 2cx(1+x^2) = 2x[c(1+x^2)] = 2xy$ ✓

Theorem. The general solution to the linear differential equation $y' + p(x)y = f(x)$ is $\boxed{y = uy_1}$ where

a) y_1 is any particular solution to the complementary equation $y' + p(x)y = 0$ and

b) $u = \int \frac{f(x)}{y_1(x)} dx$ (add a constant here).

2. Solve the IVP: $y' + 2xy = x$, $y(1) = 1$.

complementary equation: $y' + 2xy = 0$. Solution $y_1 = e^{-x^2}$

$u = \int \frac{x}{e^{-x^2}} dx = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$.

General solution: $y = e^{-x^2}\left(\frac{1}{2}e^{x^2} + c\right) = \frac{1}{2} + ce^{-x^2}$

Initial condition: $1 = y(1) = \frac{1}{2} + ce^{-1} \Rightarrow \frac{c}{e} = \frac{1}{2} \Rightarrow c = \frac{e}{2}$.

Solution: $\boxed{y = \frac{1}{2} + \left(\frac{e}{2}\right)e^{-x^2} = \frac{1}{2}\left(1 + e^{1-x^2}\right)}$

Check: $y' = -xe^{1-x^2}$ so $y' + 2xy = -xe^{1-x^2} + 2x\left[\frac{1}{2}\left(1+e^{1-x^2}\right)\right]$

$= -xe^{1-x^2} + x + xe^{1-x^2} = x$ ✓