**1.** We saw that the exponential model for population growth $P' = aP$ predicts unbounded population sizes. The logistic model resolves this problem: $P' = aP(1-\alpha P)$ where $a$ and $\alpha$ are positive constants.

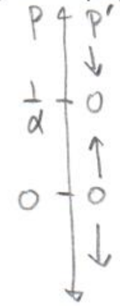
a) Draw a phase line for the model and use it to predict the long-term population trends.

$0 = P' = aP(1-\alpha P) \iff aP = 0$ or $1-\alpha P = 0$

$P<0$ doesn't make sense.

If $0 < P < \frac{1}{\alpha}$, then $P$ increases to $\frac{1}{\alpha}$

If $P > \frac{1}{\alpha}$, then $P$ decreases to $\frac{1}{\alpha}$




b) Find a solution to the differential equation using separation of variables.

$$\frac{1}{P(1-\alpha P)} P' = a.$$

$$\int \frac{1}{P(1-\alpha P)}\, dP = \int \frac{1}{P} + \frac{\alpha}{1-\alpha P}\, dP = \ln|P| - \ln|1-\alpha P| + b = \ln\left|\frac{P}{1-\alpha P}\right| + b$$

Implicit solution to the logistic equation: $\ln\left|\frac{P}{1-\alpha P}\right| = at + C.$

Solve for $P$ to get an explicit solution.

$$\left|\frac{P}{1-\alpha P}\right| = e^{at+C} = e^C e^{at} \quad \text{so} \quad \frac{P}{1-\alpha P} = k e^{at} \quad \text{where } k = \begin{cases} e^C & \text{if } \frac{P}{1-\alpha P} > 0 \\ -e^C & \text{if } \frac{P}{1-\alpha P} < 0 \end{cases}$$

$$P = ke^{at} - \alpha k P e^{at} \quad \text{so} \quad \boxed{P(t) = \frac{ke^{at}}{1+\alpha k e^{at}}}$$

If $P(0) = P_0$, then $C = \ln\left|\frac{P_0}{1-\alpha P_0}\right|$ and so $k = \frac{P_0}{1-\alpha P_0}$ using

This makes $P(t)$ $\dfrac{\frac{P_0}{1-\alpha P_0} e^{at}}{1+\left(\frac{P_0}{1-\alpha P_0}\right)\alpha e^{at}} = \dfrac{P_0}{(1-\alpha P_0)e^{-at} + \alpha P_0}$

c) Calculate $\lim_{t\to\infty} P(t)$ and compare with your answer for part a.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{ke^{at}}{1+\alpha k e^{at}} = \lim_{t\to\infty} \frac{k}{e^{-at}+\alpha k} = \frac{k}{\alpha k} = \frac{1}{\alpha}.$$

**2.** Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $T' = -k(T - T_m)$ where $k$ is a positive constant of proportionality and $T_m$ is the (constant) temperature of the environment.

a) Draw a phase line and use it to predict the eventual temperature of the object.

$0 = T' = -k(T - T_m) \iff T = T_m$

The object eventually reaches the temp. of the environment.

b) Find the general solution to the differential equation.

$T' + kT = kT_m$ linear, nonhomogeneous.

$y_1 = e^{-kt}$ $\quad u = \int kT_m e^{kt}\, dt = T_m e^{kt} + c$

$y = e^{-kt}(T_m e^{kt} + c) = T_m + c e^{-kt}$

**3.** An object with initial temperature 150°C is placed outside, where the temperature is 10°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively. At what time was the object placed outside?

Given $T_m = 10°C$. Let 12:15 be time 0.

Thus $120°C = T(0) = T_m + C = 10°C + C$ so $C = 110°C$.

Furthermore, $90 = T(5) = 10 + 110 \cdot e^{-k5}$.

Hence $\frac{80}{110} = e^{-5k}$ and so $k = -\frac{1}{5}\ln\left(\frac{80}{110}\right) \approx 0.063691$.

Now we know that $150 = T(t_0) = 10 + 110\, e^{-kt_0}$ and we can solve for $t_0$.

$$\frac{140}{110} = e^{-kt_0}$$

$$t_0 = -\frac{1}{k}\ln\left(\frac{140}{110}\right) \approx -3.7865$$

Therefore the object had been out for about 3 minutes and 47 seconds at 12:15. It must have been placed outside at about 12:11:13.

**4.** A 960 lb object is launched upward with initial velocity 60 ft/s. The atmosphere resists the object's motion with a force of 3 lb for each ft/s of speed. Assume that the only other force acting on the object is gravity (the acceleration of which is 32 ft/s$^2$ downward). Find the terminal velocity of the object.

$m = 960\text{lb}/32\text{ft}/s^2 = 30$ stones (?). From class $30v' = -30g - 3v$ and terminal velocity is $v = -\frac{30g}{3} = 320\frac{ft}{s} = 1152000\frac{ft}{hr} \approx 218.1818$ mph.