Math 260 **Laplace and the Piecewise Continuous Forcing Function** **April 4, 2018**

1. Goal: use Laplace transforms to solve the IVP: $y'' - 2y' = \begin{cases} 4, & 0 \le t < 1 \\ 6, & t \ge 1 \end{cases}$, $y(0) = -6$, $y'(0) = 1$.

a) Express $f(t) = \begin{cases} 4, & 0 \le t < 1 \\ 6, & t \ge 1 \end{cases}$ as $f(t) = f_0(t) + u(t - t_1)\,[f_1(t) - f_0(t)]$ for some functions f_0 and f_1 and a constant t_1.

b) Use your solution for part a to find the Laplace transform of $f(t)$. Use version 1 of the second shifting theorem: $L\left(u(t - t_1)g(t)\right) = e^{-t_1 s} L\left(g(t + t_1)\right)$.

c) Take the Laplace transform of the entire differential equation.

d) Sub in $L(y'') = s^2 L(y) - sy(0) - y'(0)$ and $L(y') = sL(y) - y(0)$ and solve for $Y(s) = L(y)$.

e) Take the inverse Laplace transform to get the solution $y(t) = L^{-1}(Y)$. Use the second shifting theorem: $L\left(u(t - t_1)g(t - t_1)\right) = e^{-t_1 s} L(g)$.

a) $f(t) = 4 + u(t-1)[6-4] = 4 + 2u(t-1)$.

b) $L(f) = L(4) + L(2u(t-1)) = \frac{4}{s} + e^{-s} L(2) = \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$

c,d) $s^2 L(y) - sy(0) - y'(0) - 2[sL(y) - y(0)] = \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$

Let $Y(s) = L(y)$. Also use $y(0) = -6$ and $y'(0) = 1$

$s^2 Y(s) + 6s - 1 - 2[sY(s) + 6] = (s^2 - 2s)Y(s) + 6s - 13 = \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$.

Hence $Y(s) = \frac{4 + 13s - 6s^2}{s^2(s-2)} + e^{-s}\left(\frac{2}{s^2(s-2)}\right)$

e) $L^{-1}\left(\frac{4 + 13s - 6s^2}{s^2(s-2)}\right) = L^{-1}\left(-\frac{15/2}{s} - \frac{2}{s^2} + \frac{3/2}{s-2}\right) = -\frac{15}{2} - 2t + \frac{3}{2}e^{2t}$

$L^{-1}\left(e^{-s}\left(\frac{2}{s^2(s-2)}\right)\right) = u(t-1)\,g(t-1)$ where $g(t) = L^{-1}\left(\frac{2}{s^2(s-2)}\right)$

$g(t) = L^{-1}\left(\frac{2}{s^2(s-2)}\right) = L^{-1}\left(-\frac{1/2}{s} - \frac{1}{s^2} + \frac{1/2}{s-2}\right) = -\frac{1}{2} - t + \frac{1}{2}e^{2t}$

Thus $L^{-1}\left(e^{-s}\left(\frac{2}{s^2(s-2)}\right)\right) = u(t-1)\left[-\frac{1}{2} - (t-1) + \frac{1}{2}e^{2(t-1)}\right] = u(t-1)\left(\frac{1}{2} - t + \frac{1}{2}e^{2t-2}\right)$

Solution: $y(t) = -\frac{15}{2} - 2t + \frac{3}{2}e^{2t} + u(t-1)\left(\frac{1}{2} - t + \frac{1}{2}e^{2t-2}\right)$.

Definition. Let f and g be functions such that if $t < 0$, then $f(t) = g(t) = 0$. The **convolution** of f and g is the function $f * g$ defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Theorem (Convolution Theorem).

$$L(f * g) = L(f)L(g)$$

2. Use the Convolution Theorem to evaluate the integral $\int_0^2 (2-\tau)^5\tau^7 d\tau$ by:

a) Identifying $h(t) = \int_0^t (t-\tau)^5\tau^7 d\tau$ as the convolution of two functions f and g.

b) Applying the Convolution Theorem to find $L(h)$.

c) Taking the inverse Laplace transform to find $h(t)$.

d) $\int_0^2 (2-\tau)^5\tau^7 d\tau = h(2) =?$

a) $h(t) = \int_0^t (t-\tau)^5 \tau^7 d\tau = (f * g)(t)$ for $f(t) = t^7$ and $g(t) = t^5$.

b) $L(h) = L(f*g) = L(t^7)L(t^5) = \left(\frac{7!}{s^8}\right)\left(\frac{5!}{s^6}\right) = \frac{7!\,5!}{s^{14}}$.

c) $L^{-1}\left(\frac{7!\,5!}{s^{14}}\right) = L^{-1}\left(\frac{7!\,5!}{13!}\cdot\frac{13!}{s^{14}}\right) = \frac{7!\,5!}{13!} L^{-1}\left(\frac{13!}{s^{14}}\right) = \frac{7!\,5!}{13!} t^{13}$

So $h(t) = \frac{7!\,5!}{13!} t^{13}$

d) $\int_0^2 (2-\tau)^5 \tau^7 = h(2) = \frac{7!\,5!}{13!} 2^{13} = \frac{1024}{1287} \approx 0.79565$