Theorem. Let A be an $n \times n$ matrix with real entries.

i) If A has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with associated **linearly independent** eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then the functions

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t} \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_2 t} \\ &\vdots \\ \mathbf{y}_n &= \mathbf{x}_n e^{\lambda_n t} \end{aligned}$$

form a fundamental set of solutions of $A\mathbf{y} = \mathbf{y}'$.

ii) If A has an eigenvalue λ with multiplicity of 2 or more and with an associated eigenspace of dimension 1, then there are infinitely many vectors $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{x}$. If $\mathbf{u}$ is any such vector, then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x} e^{\lambda t} \\ \mathbf{y}_2 &= \mathbf{x} t e^{\lambda t} + \mathbf{u} e^{\lambda t} \end{aligned}$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.

a) If λ has multiplicity 3 or more and has an associated eigenspace of dimension 1, then in addition to the preceding there are infinitely many vectors $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{u}$. If $\mathbf{v}$ is any such vector, then an additional linearly independent solution is

$$\mathbf{y}_3 = \mathbf{v} e^{\lambda t} + \mathbf{u} t e^{\lambda t} + \mathbf{x} \left(\frac{t^2}{2} \right) e^{\lambda t}$$

b) If λ has multiplicity 3 or more and has an associated eigenspace of dimension 2, then see page 550 of the textbook for a solution.

iii) If A has a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, then both $\mathbf{u}$ and $\mathbf{v}$ are nonzero and

$$\begin{aligned} \mathbf{y}_1 &= e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \\ \mathbf{y}_2 &= e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned}$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.

Method. To solve the IVP $A\mathbf{y} = \mathbf{y}'$, $\mathbf{y}(0) = \mathbf{b}$:

1. Find the eigenvalues of A.
2. Find an eigenvector for each eigenvalue.
3. Use the theorem above to find a fundamental set of solutions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$.
4. Your general solution is $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$.
5. Use the initial condition to solve the system of equations for $c_1, c_2, \ldots, c_n$:
   $\mathbf{b} = \mathbf{y}(0) = c_1\mathbf{y}_1(0) + c_2\mathbf{y}_2(0) + \cdots + c_n\mathbf{y}_n(0)$.

Method. The eigenvalues of an $n \times n$ matrix A are the solutions to $\det(A - \lambda I) = 0$. Find the eigenvector(s) corresponding to eigenvalue λ by solving $(A - \lambda I)\mathbf{x} = 0$. In 2-dimensional systems, the solution to $(A - \lambda I)\mathbf{x} = 0$ is usually a line $ax_1 + bx_2 = 0$; to find an eigenvalue, just choose a value for x_1 or x_2 and solve for the other..