Exam 1

Thoerem. *The general solution to the homogeneous linear differential equation* $y' + p(x)y = 0$ *is*

$$\boxed{y = ce^{-P(x)}}$$

where $P'(x) = p(x)$.

Thoerem. *The general solution to the linear differential equation* $y' + p(x)y = f(x)$ *is* $\boxed{y = uy_1}$ *where*

a) y_1 *is any particular solution to the complementary equation* $y' + p(x)y = 0$ *and*

b) $u = \displaystyle\int \frac{f(x)}{y_1(x)}dx$ *(add a constant here).*

Conversion to the Ponicaré phase plane: $y' = v$ and $y'' = v\dfrac{dv}{dy}$.

Exam 2

Thoerem. *If the characteristic polynomial of* $ay'' + by' + cy' = 0$ *has...*

a) ...distinct real roots r_1 *and* r_2, *then a general solution is* $y = c_1e^{r_1x} + c_2e^{r_2x}$

b) ...a single (repeated) real root r, *then a general solution is* $y = e^{rx}(c_1 + c_2x)$

c) ...complex conjugate roots $\lambda \pm i\omega$ *(where* $\omega > 0$*), then a general solution is* $y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$

Thoerem. *If* y_p *is any particular solution to the differential equation* $y'' + p(x)y' + q(x)y = f(x)$ *and* $\{y_1, y_2\}$ *is a fundamental set of solutions to the complementary equation, then a general solution for differential equation is*

$$y = y_p + c_1y_1 + c_2y_2$$

Thoerem (Superposition)**.** *If* y_{p_1} *is a particular solution of* $y'' + p(x)y' + q(x)y = f_1(x)$ *and* y_{p_2} *is a particular solution of* $y'' + p(x)y' + q(x)y = f_2(x)$, *then a particular solution of*

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

is

$$y_p = y_{p_1} + y_{p_2}$$

Method. To find a particular solution for $ay'' + by' + cy = e^{\alpha x}G(x)$ (where G is a polynomial), solve for the coefficients of $Q(x)$ in the following (where Q is a polynomial with the same degree as G):

a) $y_p = e^{\alpha x}Q(x)$ if $e^{\alpha x}$ is not a solution to the complementary equation;

b) $y_p = xe^{\alpha x}Q(x)$ if $xe^{\alpha x}$ is not a solution to the complementary equation, but $e^{\alpha x}$ is a solution to the complementary equation;

c) $y_p = x^2e^{\alpha x}Q(x)$ if $xe^{\alpha x}$ and $e^{\alpha x}$ are both solutions to the complementary equation.

Method. To find a particular solution for $ay'' + by' + cy = P(x)\cos \omega x + Q(x)\sin \omega x$ (where P and Q are polynomials), solve for the coefficients of $A(x)$ and $B(x)$ in the following (where A and B are polynomials with degree equal to the larger of the degrees of P and Q):

a) $y_p = A(x)\cos \omega x + B(x)\sin \omega x$ if $\cos \omega x$ and $\sin \omega x$ are not solutions to the complementary equation;

b) $y_p = x\,[A(x)\cos\omega x + B(x)\sin\omega x]$ if $\cos\omega x$ and $\sin\omega x$ are solutions to the complementary equation;

Power Series Solutions

Method (Power series solutions for $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$)**.** If $P_0(x_0) \neq 0$, then we'll say that x_0 is an **ordinary point** of the differential equation. We seek a power series solution centered at x_0:

$$\boxed{y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots}$$

Using these in our differential equation gives recursive formulas for the coefficients. Note that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Thus, inital values for $y(x_0)$ and $y'(x_0)$ allow us to find values for all the coefficients.

Comment. The method works at any ordinary point, but we'll usually work with $x_0 = 0$.
