Method. Let A be an 2×2 matrix: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. To solve the IVP $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{b}$:

1. Find the eigenvalues of A.
2. Find an eigenvector for each eigenvalue.
3. Use the theorem below to find a fundamental set of solutions $\mathbf{y}_1, \mathbf{y}_2$.
4. Your general solution is $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.
5. Use the initial condition to solve the system of equations for c_1, c_2: $\mathbf{b} = \mathbf{y}(0) = c_1\mathbf{y}_1(0) + c_2\mathbf{y}_2(0)$.

Definition. The matrix I is the **identity matrix** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$ and $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$.

Method. The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$ (solve for λ, often using the quadratic formula). Find the eigenvector(s) corresponding to each eigenvalue λ by solving $(A - \lambda I)\mathbf{x} = 0$ (sub in the eigenvalue and solve for the two entries in $\mathbf{x}$). In 2-dimensional systems, the solution to is usually a line; to find an eigenvalue, just choose a non-zero value for x_1 or x_2 and solve for the other.

Theorem. Let A be an 2×2 matrix. To find a fundamental set of solutions to $\mathbf{y}' = A\mathbf{y}$:

i) If A has real eigenvalues λ_1, λ_2 with associated **linearly independent** eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t}$$
$$\mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t}$$

form a fundamental set of solutions of $A\mathbf{y} = \mathbf{y}'$.

ii) If A has an eigenvalue λ with multiplicity of 2 or more and with an associated eigenspace of dimension 1 and $\mathbf{x}$ is any eigenvector, then there are infinitely many vectors $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{x}$. If $\mathbf{u}$ is any such vector, then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda t}$$
$$\mathbf{y}_2 = \mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.

iii) If A has a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, then both $\mathbf{u}$ and $\mathbf{v}$ are nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t)$$
$$\mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)$$

are linearly independent solutions of $A\mathbf{y} = \mathbf{y}'$.